Revealing Private Information in a Patent Race*

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Abstract

This article investigates the role of private information in patent races. Although prior work assumes that firms observe their rivals' progress, R&D is often conducted in secrecy. We analyze how the race dynamics change when progress is private and examine whether voluntary disclosure is strategically beneficial, even without direct payoff consequences. We show that a firm may disclose its breakthrough to discourage a rival's R&D effort, but only when the rival has not yet done so and R&D efficiency is sufficiently low. The unique equilibrium takes one of three forms: no-revelation, instant-revelation, or mixed-revelation.

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1 Introduction

In practice, firms conduct their R&D largely in secrecy. In contrast, patent race literature has typically assumed that firms' progress is publicly observable. In this article, we relax this assumption and analyze a patent race in which the firm's progress is its private information. Departing from the assumption of complete information significantly affects the dynamics of the patent race. Lacking information on rival progress, firms can only act based on their beliefs. As their beliefs evolve, firms continually adjust the intensity of their R&D investments. This raises several questions. How does a firm's R&D effort evolve over the course of the race? Do firms invest more intensely early on, or do they invest increasingly aggressively over time? How does a firm's R&D effort change with the arrival of a breakthrough? Most importantly, do firms have strategic incentives to reveal their progress voluntarily?

We study a private information version of the patent race introduced in Grossman and Shapiro (1987). There are two firms that compete in making a patentable discovery. At any point in time, the firms continuously adjust their research efforts, which affect the Poisson arrival rates of breakthroughs and involve flow costs. In order to obtain the final discovery, each firm needs to attain two consecutive breakthroughs. Obtaining the first (interim) breakthrough does not directly affect payoffs. Only winning the patent race by attaining two breakthroughs yields a positive payoff and also ends the game.

A novel feature of the model is that a firm's first breakthrough is private information unless voluntarily disclosed. The disclosure is costless, verifiable, and has a purely informational effect with no technological spillovers, no licensing, and no intermediate rewards. Thus, the only motive for revealing the breakthrough is strategic: to discourage the rival's investment. We think about the first breakthrough as a prototype of a new technology or results of first trials in pharmaceutical research. A pharmaceutical firm can publish results of audited randomized trials, without publishing details about the drug. A technological firm can demonstrate a prototype, as Samsung did with its folding smartphone, or SpaceX with its rocket launches, publicly demonstrating capabilities without disclosing the underlying technology.

We analyze four different information settings with increasing complexity. As a benchmark, we first solve the complete information version of the patent race, where the interim breakthrough is observed by the rival. We show that when a rival achieves a breakthrough, a firm with a breakthrough increases effort, whereas a firm without a breakthrough reduces effort. This aligns with standard findings in studies of complete information patent races, where firms exert the highest effort when the race is neck-and-neck (Grossman and Shapiro, 1987; Harris and Vickers, 1987).

Second, we study the private information setting where each firm only observes its own progress, without an option to reveal it. As time proceeds, each firm updates its belief about the rival having a breakthrough. This involves two opposing effects: a positive effect due to the rival becoming increasingly likely to have completed the first stage of R&D over time and a negative effect due to not observing the rival patenting the final discovery. Indeed, if the rival had already completed the first stage of R&D, the more time that has passed, the more likely it would have completed the second stage as well. We show that the first effect dominates the second one so that the posterior probability that the rival has completed the first stage (conditional on neither firm having patented by time t) keeps increasing over time. Moreover, it converges to a critical point value that is strictly smaller than 1 due to the second effect. As time proceeds, a firm without a breakthrough becomes increasingly convinced that its rival leads the race and decreases its effort over time. However, once a firm completes the first stage of R&D, its effort jumps upwards and continues to increase as it becomes more likely that the race is tight.

Third, we consider an asymmetric version of the private information setting, where one of the two firms (say, firm A) is known to have achieved a breakthrough. Although such a situation is interesting on its own, it is essential for further analysis of firms' decisions to reveal breakthroughs. Much like in the previous scenario, firm A increases its effort with increasing belief about firm B's progress. Firm B, although its belief does not change, responds by increasing its effort as well, escalating competitive actions. In addition, once firm B achieves a breakthrough, knowing that the race is tied, its effort exceeds that of firm A. Consequently, this puts firm A at a disadvantage relative to firm B.

Fourth, we study the patent race with the option to disclose breakthroughs. Our first result is that a firm never discloses a breakthrough knowing that its rival has achieved a breakthrough. In fact, a rival that has achieved a breakthrough becomes only encouraged to work harder by learning about the firm's breakthrough. Consequently, if one firm discloses a breakthrough, the other firm will conceal its progress until achieving the second breakthrough, yielding the asymmetric case described above. Knowing that, we analyze firms' strategies before any of them has revealed a breakthrough. This is a symmetric situation. Revelation of a firm's (say firm A's) breakthrough involves three effects on the rival's (firm B's) effort: The desirable effect is that it discourages the rival if the rival has not yet achieved a breakthrough. However, there are two undesirable effects that come with the revelation. First, if the rival has already achieved a breakthrough, or once it does so, having the information about firm A's breakthrough causes it to increase effort. Second, by revealing its breakthrough, firm A is put at an informational disadvantage, as firm B, once informed, will withhold its own status.

Depending on the magnitude of these effects, we identify three possible types of equilibria. In the *no-revelation equilibrium*, firms never disclose their first breakthrough. In the *instant-revelation equilibrium*, firms immediately reveal their first breakthrough. The third equilibrium type involves firms adopting mixed strategies, revealing their breakthroughs only with a certain probability. This equilibrium has a distinct structure: firms randomize their decisions only up to a certain deadline, after which they cease revealing their breakthroughs altogether. We refer to this as the *mixed-revelation equilibrium*.

Due to the complexity of the dynamic system, directly comparing the effects of revelation or determining the equilibrium type for specific parameter values is infeasible. To address this, we employ numerical simulations. The simulations reveal that the equilibrium is always unique. Specifically, the *instant-revelation equilibrium* arises when research is difficult (or inefficient), meaning that achieving a breakthrough is a long-term process. In such cases, rivals are likely to remain unsuccessful for an extended period, allowing the desirable effect of discouraging an unsuccessful rival to outweigh any undesirable effects. Conversely, when research is easy (or efficient), firms anticipate that their rivals will quickly catch up. As a result, they withhold their initial breakthroughs, leading to a *no-revelation equilibrium*. Finally, when research is moderately difficult, successful firms are inclined to disclose their progress but may wait for their rivals to reveal their breakthroughs first. Consequently, firms randomize their decisions about revelation, resulting in a *mixed-revelation equilibrium* akin to those observed in war of attrition games.

The structure of this article is as follows. Following a literature review, Section 2 introduces the model of the patent race. Section 3 analyzes the complete information version of the game as a benchmark. Section 4 introduces the patent race with private information and develops the framework used in the subsequent two sections: Section 5 analyzes firms' R&D effort over time in the symmetric case where both firms' states are unknown, and Section 6 explores an asymmetric version of the model, where one firm's breakthrough is already publicly known. Section 7 investigates scenarios where firms can choose to disclose their breakthroughs and provides a characterization of the equilibrium. Section 8 concludes. The proofs of all propositions and lemmas are relegated to the Appendix or the Supplementary Appendix.

Related Literature

The study of R&D investments in a competitive environment was pioneered by Loury (1979), Lee and Wilde (1980), and Dasgupta and Stiglitz (1980). In those seminal studies, the patent race is characterized as a static game, where firms choose their R&D efforts at the outset. This analysis was later expanded to dynamic settings by Grossman and Shapiro (1987) and Harris and Vickers (1987), who assumed a definitive finish line. In their framework, a firm wins the race by completing a specified number of R&D stages. A primary result of these models is that firms invest in R&D most intensively when they are neck-and-neck and near the finish line. Conversely, Hörner (2004) examines a perpetual race where the leading firm receives a continuous flow of payoffs. His findings contrast with earlier models, demonstrating that the fiercest competition does not necessarily occur when firms are closest to one another.

The study most closely related to ours is Gordon (2011), which also examines a two-stage patent race with private information. However, Gordon's framework restricts firms to binary effort levels (high or low). Although this restriction significantly simplifies the model's analysis, it also has two crucial limitations. First, because effort cannot be adjusted incrementally, information revelations in Gordon (2011) only impact the race's dynamics when they are sufficiently large to trigger a jump between the two effort levels. Second, the discrete effort choice space can lead to multiplicity of equilibria depending on parameter values, a complexity not present in our model. Specifically, Gordon (2011) identifies only two types of equilibria that resemble our pure-strategy equilibria: a non-revealing equilibrium, which exists across all relevant parameter values, and a partially revealing equilibrium, which coexists with the former only under specific conditions. In contrast, our continuous framework ensures a unique equilibrium and characterizes a mixed-revelation type, a third strategic outcome that the discrete structure of Gordon (2011) cannot fully capture.

A similar question to ours—namely whether to disclose an intermediate innovation—is also studied by two closely related papers, Chatterjee, Das, Dong, and Klein (2026) and our companion working paper Kocourek and Kováč (2026). Chatterjee et al. (2026) consider a sequential race with two stages and fixed rewards for being the first to complete and disclose each stage, whereas Kocourek and Kováč (2026) assume that firms conduct R&D on two products simultaneously and focus on how the equilibrium depends on the underlying market structure, namely, whether products are substitutes or complements. Both studies differ from ours by assuming that disclosure directly affects payoffs and triggers technological spillovers. Despite these differences, both identify three types of equilibria that partially overlap with our results. Specifically, their pure-strategy equilibria correspond to our *instant-revelation* and *no-revelation* equilibria. However, their mixed-strategy equilibria exhibit an inverse temporal structure compared to that of our *mixed-revelation equilibrium*. In our model, firms disclose with a positive hazard rate from the moment of breakthrough but eventually cease randomization, resembling a war of attrition game. In contrast, in Chatterjee et al. (2026) and Kocourek and Kováč (2026), firms initially refrain from disclosing and only begin randomizing with a certain probability after a specific deadline has passed, essentially transforming the disclosure decision into a preemption game.

Several studies explore alternative motivations for strategic disclosure in the context of patent races. Kim and Poggi (2025) analyze a model with two R&D paths, where firms may patent interim success to signal rivals to revise their strategies. They identify conditions for a patenting equilibrium (disclosure) and a concealment equilibrium (withholding the innovation). Song and Zhao (2021) examine a two-stage game where disclosure serves as a "proof of concept" regarding innovation feasibility. They demonstrate that the equilibrium exhibits a "disclose-withhold-exit" pattern: withholding success makes rivals pessimistic enough to eventually exit the race. Additionally, Gill (2008) investigates disclosure as a signaling tool to deter rival investment in the presence of technological spillovers. Other studies compare patenting versus secrecy in the context of protection of property rights when facing imitation threats (Anton and Yao, 2004) or technological spillovers (Kultti, Takalo, and Toikka, 2007).

Several studies analyze innovation races with private information, where firms' strategies involve optimal stopping times rather than continuous effort. Hopenhayn and Squintani (2016) consider a sequence of patent races and assume that the value of innovation grows deterministically after a breakthrough. They identify conditions under which firms patent suboptimally too early or too late in equilibrium. Bobtcheff, Bolte, and Mariotti (2017) study the trade-off between the risk of preemption and quality of innovation. They argue that when breakthroughs become more frequent, competitive pressure leads to premature patenting of low-quality innovations. Bobtcheff, Lévy, and Mariotti (2025) explore the tension between preemption concerns and the possibility of learning from others. They identify a version of the winner's curse, which causes firms to delay investments in order to acquire more information.

The role of private information has also been studied in other dynamic game frameworks. In a series of papers, P. Strack and various coauthors examine the role of private information in general stopping-time games, where the primary strategic decision involves the optimal timing for exiting or terminating a process (Seel and Strack, 2016; Kruse and Strack, 2015; Fudenberg, Strack, and Strzalecki, 2018). Within the literature on strategic experimentation and contests, several studies analyze how private information shapes investment. In the framework of strategic experimentation, Heidhues, Rady, and Strack (2015) investigate strategic experimentation with private payoffs and Halac, Kartik, and Liu (2017) analyze optimal information design within competitive contests. Furthermore, Achim and Klein (2022) explore strategic investment under private information and Hoppe-Wewetzer, Katsenos, and Ozdenoren (2023) compare public and private learning in the context of scientific rivalry. Finally, Bonatti and Hörner (2011) study collaborative experimentation in a setting with linear costs, where bang-bang solutions apply.

2 Model

We study an infinite-horizon continuous-time model of a patent race with two risk-neutral firms A and B. The firms invest in R&D to develop innovations that may earn them a specific patent. In order to patent the discovery, a firm must achieve two consecutive well-defined breakthroughs. We define the state of firm $j \in \{A, B\}$ as the number of breakthroughs it has achieved by time t, denoted by $k_t^j \in \{0, 1, 2\}$. Initially, each firm is in state 0. Once a firm makes the first breakthrough, it reaches state 1. Upon achieving the second breakthrough, the firm reaches state 2, secures the patent, and realizes its value, $v > 0$, at which point the game ends. The payoff v is the only positive payoff a firm can obtain in our model.

To obtain the breakthrough, the firms conduct R&D. At any time $t \geq 0$, each firm $j \in \{A, B\}$ chooses its research effort $e_t^j \geq 0$. We characterize this effort as the instantaneous hazard rate of a breakthrough at time t. This implies that knowledge is not accumulated unless a breakthrough is achieved; this process can be interpreted as consisting of independent trials. Obtaining a breakthrough within the time interval $[t, t+\Delta t]$ results in the transition of state k_t^j at time t to $k_{t+\Delta t}^j = k_t^j + 1$ at time $t + \Delta t$, which happens with the probability[1,2]

$$\Pr[k_{t+\Delta t}^j = k_t^j + 1] = e_t^j \Delta t + o(\Delta t).$$

The research effort, driven by R&D investments, involves costs. Firm j incurs a flow cost $c(e_t^j)$ that depends only on its current effort. We assume that the cost function is convex, which implies that the marginal cost of effort is increasing. For tractability, we assume that the cost function takes a quadratic form,[3]

$$c(e) = \tfrac{1}{2} a e^2,$$

where $a > 0$ is a parameter. The assumption of quadratic costs is restrictive, but it simplifies the algebra and the equilibrium characterization. All future payoffs are discounted at a constant rate $r > 0$. Let τ^j denote the (random) time at which firm j files a patent (or infinity) and let $\tau = \min\{\tau^A, \tau^B\}$. The realized payoff of firm j is

$$\Pi^j = \underbrace{-\int_0^{\tau^j} \exp(-rt) \cdot c(e_t^j) dt}_{\text{effort cost}} + \underbrace{\exp(-r\tau^j) \cdot v \cdot \mathbf{1}_{\tau=\tau^j}}_{\text{value of the patent}}.$$

The first term (effort cost) represents the accumulated flow costs of research. Firm j bears

[1]We assume that each realization of the trajectory $t \mapsto k_t^j$ as well as $t \mapsto e_t^j$ is right-continuous.
[2]$o(\cdot)$ represents any function such that $o(\Delta t)/\Delta t \to 0$ as $\Delta t \searrow 0$.
[3]Superscript 2 indicates an exponent, whereas 0 or 1 refer to states.

those costs, regardless of whether it wins the patent race. The second term represents the patent's value v, discounted to the present value at time τ, applicable only when firm j wins the patent race. Each firm maximizes its expected profit $\mathrm{E}[\Pi^j]$.[4]

The model involves only three parameters: the value of the patent v, the effort cost parameter a, and the discount rate r. These parameters are assumed to be commonly known to the firms. Without loss of generality, we may set $v = 1$ and $a = 1$. Indeed, apart from choosing a unit of value such that $v = 1$, it is also possible to choose the units of time such that $a = 1$. In that respect, all results can be formulated in terms of a single parameter $\rho = ar/v$, which we call the *research difficulty*. Its inverse, $1/\rho = v/(ar)$, can then be interpreted as the *R&D efficiency*.[5]

3 Benchmark: Patent Race with Observable Progress

As a benchmark, we analyze the complete information version of the patent race in which firms' progress is common knowledge; specifically, each firm knows its rival's state.[6] We restrict attention to symmetric Markov perfect equilibria.[7] The state of the game is the pair $(k, l) \in \{0, 1, 2\}^2$, where k denotes firm A's state and l denotes firm B's state. Because the model features neither private information nor learning, firms' efforts and continuation values remain constant (under the Markov assumption) until a new breakthrough occurs. Denote by $v_C^{k,l}$ and $e_C^{k,l}$ the continuation value and effort of a firm in state k when its rival is in state l.[8] Because the race ends once a firm achieves its second breakthrough, we set $v_C^{2,l} = v$ for the winner of the patent race and $v_C^{k,2} = 0$ for

[4] We use the variable e for effort, whereas $\exp(\cdot)$ denotes the exponential function. The function $\mathbf{1}_{\tau=\tau^j}$ is the indicator function determining whether firm j wins the race. Note that the case of both firms patenting simultaneously occurs with zero probability and can thus be neglected.

[5] Although we keep the parameters v and a in the main text, we will use the normalization $v = 1$ and $a = 1$ in the proofs in the Appendix. Furthermore, we will use the normalization in numerical computations, where it reduces the dimension of the relevant parameter space to one.

[6] This model has been analyzed in Harris and Vickers (1987) under the assumption $r = 0$, or in Grossman and Shapiro (1987), which considers a more general class of cost functions and therefore produces less conclusive results.

[7] In fact, the game admits only a symmetric equilibrium, as can be shown with a minor modification to the proof of uniqueness. We impose symmetry here for notational simplicity.

[8] The subscript C stands for complete information.

its rival (where $k, l \in \{0, 1\}$).

We relegate the details of the analysis to Appendix A. Nevertheless, let us point out that the optimal effort in state (k, l) satisfies the first order condition

$$ae_C^{k,l} = v_C^{k+1,l} - v_C^{k,l}, \tag{1}$$

which implies that the optimal effort is directly proportional to the potential gains from making a breakthrough. The above condition can also be interpreted as an equation between the marginal cost of effort and marginal benefit: the left-hand side represents the instantaneous marginal cost of effort, whereas the right-hand side reflects the expected marginal benefit, i.e., the increase in continuation value from transitioning to the next state. We derive analogous conditions in other versions of the model.

Under the optimal efforts, as given by condition (1), we obtain a system of four equations (one for each state) with the remaining continuation values (v_C^{00}, v_C^{10}, v_C^{01}, and v_C^{11}) as unknowns. The following proposition asserts that the system has a unique solution, which allows us to compare firms' research efforts among different scenarios.[9]

Proposition 1. *In a patent race with complete information, there is a unique symmetric Markov perfect equilibrium. The equilibrium effort levels satisfy:*

$$\text{(i)} \quad e_C^{01} < e_C^{00} \qquad \text{and} \qquad \text{(ii)} \quad e_C^{10} < e_C^{11}.$$

Both inequalities compare the efforts exerted when a rival achieves a breakthrough. The first inequality shows that a rival's breakthrough discourages the firm without a breakthrough from exerting effort. The second inequality shows that a firm with a breakthrough is motivated to increase its effort in response to its rival's breakthrough. These findings align with earlier literature (Grossman and Shapiro, 1987; Harris and Vickers, 1987), which concluded that firms exert higher effort when the race is tight.

[9] Grossman and Shapiro (1987) also show that $e_C^{11} > e_C^{10} > e_C^{01}$. However, because they allow for a more general class of convex effort cost functions, they conclude that the relationship between e_C^{00}, e_C^{01}, and e_C^{10} is ambiguous.

Based on the above results, we can expect that in the private information setting, in which a firm cannot observe a rival's state and must rely solely on updated posterior beliefs about its rival, an unsuccessful firm will progressively reduce its effort as it becomes increasingly convinced that its rival leads. Conversely, a successful firm will gradually increase effort over time, perceiving the competition as increasingly neck-and-neck. Furthermore, these findings illuminate the strategic decision of whether firms should disclose their successes when having the option to do so credibly without leaking any technological secrets. In particular, they suggest that a firm would reveal its breakthrough only if it expects to discourage the rival's R&D efforts. This occurs when the firm believes its rival remains unsuccessful and is unlikely to catch up soon.

4 Patent Race with Unobservable Progress

We now model a patent race in which each firm's progress is private information. Each firm knows its own progress towards the patent, but cannot observe the rival's progress. In this version of the model, we assume that there is no mechanism for voluntary disclosure. Because attaining the second breakthrough ends the game, we assume that firm j privately knows whether it is in state 0 or state 1, that is, whether $k_t^j = 0$ or $k_t^j = 1$. Moreover, firm j's effort remains unobservable to the rival so the rival cannot infer k_t^j from observed effort.

Given that a firm does not know the rival's state, it forms a belief about it. Define p_t^j as the probability, from firm $-j$'s perspective, that firm j is in state 1 (where $j \in \{A, B\}$):[10]

$$p_t^j = \Pr[k_t^j = 1 \mid k_t^j < 2]. \tag{2}$$

This specification of beliefs yields non-trivial dynamics. Although the unconditional probability of having a breakthrough steadily increases over time, the dynamics of beliefs is now more complex. As the game ends once a firm attains state 2, firm $-j$ knows

[10]We adopt the standard notation in which $-j$ refers to firm j's rival.

with certainty that $k_t^j < 2$ as long as the race is still ongoing. Thus, firm $-j$ also conditions its belief on the fact that firm j has not yet attained its second breakthrough, specifically that $k_t^j < 2$. As time proceeds, both firms continuously update their beliefs in accordance with Bayes' law. If we denote by $e_t^{k,j}$ the effort of firm j when in state $k \in \{0, 1\}$ at time t, the posterior belief is then governed by the well-known law of motion (see, e.g., Keller, Rady, and Cripps, 2005) specified in the following lemma. Its proof is provided in Supplementary Appendix B.

Lemma 1. *The posterior belief* p_t^j *follows the law of motion* $\dot{p}_t^j = (1 - p_t^j)(e_t^{0,j} - p_t^j e_t^{1,j})$.

Intuitively, a higher effort in state 0 makes the first breakthrough more likely. At the same time, a higher effort in state 1 makes the second breakthrough more likely. More precisely, the evolution of beliefs is determined by two key factors: the hazard rate of transitioning to state 1 and the hazard rate of advancing from state 1 to the final patent. The former is equal to the effort in state 0, which is $e_t^{0,j}$. The latter is equal to $p_t^j e_t^{1,j}$, as from the perspective of the rival, firm j has made the first breakthrough with probability p_t^j, and if that is the case, then it patents with the hazard rate $e_t^{1,j}$.

Note that once the belief p_t^j reaches the value 1, it remains constant thereafter. One might expect that p_t^j will eventually approach 1 as time progresses. However, as long as $p_0^j < 1$, this is not the case. Although the probability that the rival has made the first breakthrough increases over time, so too does the probability that the rival, having potentially achieved the second breakthrough, has already patented and thus exited the game. Therefore, as we formalize below, conditioning on the rival not having patented, the probability that it is in state 1 asymptotically converges to a value strictly less than 1.

We focus on *Markov perfect Bayesian equilibria (MPBE)* of the game. Firms condition their actions on the payoff-relevant state, which (for firm j) consists of its own state k_t^j and the profile of mutual posterior beliefs (p_t^A, p_t^B). Firm j's strategy involves selecting an effort $e \geq 0$ which is a function of its own state and the mutual posterior beliefs, expressed as a function of the triple (k_t^j, p_t^A, p_t^B). To simplify notation, denote firm j's effort in state $k \in \{0, 1\}$ at time t by $e_t^{k,j}$, given that the pair of beliefs (p_t^A, p_t^B) depends only on time.[11] It is crucial to note that the choice of effort does not depend on actual calendar time, and thus $e_{t_1}^{k,j} = e_{t_2}^{k,j}$, whenever $(p_{t_1}^A, p_{t_1}^B) = (p_{t_2}^A, p_{t_2}^B)$.

As in the previous section, every state has an associated continuation value. We denote it by the parallel notation $v_t^{k,j}$. As in the benchmark case, $v^{2,j} = v$ for the winner and 0 for its rival. In Appendix B.1 we derive for each value function $v_t^{k,j}$ (where $k \in \{0,1\}$ and $j \in \{A,B\}$) a corresponding ordinary differential equation (ODE).[12] Jointly with the law of motion for each firm's beliefs, we obtain a system of six ODEs that govern the dynamics of the model.

The analysis of the private information game proceeds in two steps. We first consider the symmetric case in which both firms start in state 0 and neither firm's progress is known to its rival (Section 5). We then turn to the asymmetric case in which one firm is publicly known to have achieved a breakthrough while its rival's state remains private (Section 6). The latter case also serves as the continuation game following a disclosure event in the model with optional revelation analyzed in Section 7.

5 Patent Race with Both Firms in Unknown State

In this section we focus on symmetric equilibria of the private information model in which both firms start from the state $k_0^j = 0$. We show the existence and uniqueness of such equilibrium, and we study its properties such as the dynamics of firms' R&D efforts over time.

Due to the symmetry assumption, both firms have the same strategies and the same posterior beliefs $p_t^A = p_t^B = p_{S,t}$ with $p_{S,0} = 0$. In this setting, we denote player j's effort when in state $k \in \{0,1\}$ at time t as $e_{S,t}^k$ and the corresponding continuation value as $v_{S,t}^k$.[13] Furthermore, symmetry reduces the system of six ODEs (see (24)–(26) in

[11] Note that firm j's effort only depends on its own state, as opposed to the complete information case, where the effort depends on both firms' states.

[12] This derivation follows the standard procedure, expressing the difference of value functions $v_t^{k,j} - v_{t+\Delta t}^{k,j}$ at times t and $t + \Delta t$. Dividing by Δt and taking the limit $\Delta t \searrow 0$ yields the corresponding ODE for state k.

[13] Because both firms adopt identical strategies in the symmetric equilibrium, we drop the firm index

Appendix B.1) into the following system of three ODEs:[14]

$$-\dot{v}^1_{S,t} = \frac{a}{2}(e^1_{S,t})^2 - (r + p_{S,t}e^1_{S,t})v^1_{S,t} \tag{3}$$

$$-\dot{v}^0_{S,t} = \frac{a}{2}(e^0_{S,t})^2 - (r + p_{S,t}e^1_{S,t})v^0_{S,t} \tag{4}$$

$$\dot{p}_{S,t} = (1 - p_{S,t})(e^0_{S,t} - p_{S,t}e^1_{S,t}), \tag{5}$$

along with the identities

$$ae^1_{S,t} = v - v^1_{S,t}, \qquad ae^0_{S,t} = v^1_{S,t} - v^0_{S,t}, \tag{6}$$

the initial condition $p_{S,0} = 0$, and the constraints $v^0_{S,t}, v^1_{S,t} \in (0, v)$, $v^0_{S,t} < v^1_{S,t}$, and $p_{S,t} \in [0, 1)$, for all $t \geq 0$. To satisfy the Markov condition, efforts $e^k_{S,t}$ as well as continuation values $v^k_{S,t}$ can depend on time only via the belief $p_{S,t}$. More precisely, there exist time-independent functions E^k_S and V^k_S such that $e^k_{S,t} = E^k_S(p_{S,t})$ and $v^k_{S,t} = V^k_S(p_{S,t})$ for any $k \in \{0, 1\}$ and $t \geq 0$.

As in the benchmark case, the conditions (6) represent equations between marginal cost of effort and marginal benefit of transition to the next state. Note that the ODEs (3)–(4) feature the term $r + p_{S,t}e^1_{S,t}$, representing the risk-adjusted discount rate. This is composed of the discount rate r and the perceived hazard rate of the rival's patenting, which is $p_{S,t}e^1_{S,t}$.

Solving the system of ODEs (3)–(5) is complicated by the fact that it is not an initial value problem: whilst we know $p_{S,0} = 0$, we do not know the initial values of $v^1_{S,0}$ and $v^0_{S,0}$, and an error of the initial guess grows exponentially going forward in time. Conversely, solving the equations backwards in time would let the error in the guess of $p_{S,t}$ grow exponentially. Despite these challenges, we are able to demonstrate the uniqueness of the solution.

Proposition 2. *The patent race with private information has a unique symmetric Markov*

j for notational simplicity.

[14]We use subscript S to denote the symmetric equilibrium.

perfect Bayesian equilibrium.

The proof of the result is provided in Appendix C. The proof proceeds in three steps, each being formulated as a separate lemma. First, we show that the system of ODEs has a unique critical point (Lemma C.1). Second, we show that there is a unique direction in which a solution can converge to this point (Lemma C.2). Third, we show that every solution has to converge to the critical point (Lemma C.3).

Knowing that there is a unique equilibrium, we can discuss its properties. The following proposition provides results about the dynamics of firms' beliefs and efforts (see Appendix C for a proof).

Proposition 3. *In the equilibrium of the patent race with private information, the belief* $p_{S,t}$ *is increasing over time and converges to the value* $p_{S,*}$ *such that* $p_{S,*} < 1$*, where* $p_{S,*}$ *is the belief at the critical point. Moreover, each firm's effort is decreasing over time until the firm makes the first breakthrough. Subsequently, the effort jumps upwards and continues to rise after the breakthrough.*

The proposition consists of four statements. First, the belief increases over time and converges to a value smaller than 1. The monotonicity is a consequence of the Markov property. This implies that, as time proceeds, each firm is increasingly likely to have a breakthrough. However, at the same time the firm is increasingly likely to have patented the second one already. Thus, conditioning on the fact that the rival did not win the race yet, the belief $p_{S,t}$ remains bounded away from certainty. Second, a successful firm gets increasingly rivalrous, i.e., the effort $e^1_{S,t}$ increases over time. This is quite intuitive as the rival is increasingly likely to be successful over time, which means that the race is more likely neck-and-neck. Third, an unsuccessful firm gives up over time, i.e., $e^0_{S,t}$ decreases over time. As time proceeds, the firm without any breakthrough believes it is more likely to be behind. As a response, it decreases its effort. Fourth, a successful firm is always more rivalrous than an unsuccessful one, i.e., $e^1_{S,t} > e^0_{S,t}$. Intuitively, the firm's marginal benefit from effort increases after a breakthrough, whereas the belief about the rival remains the same. Thus, it pays off to increase the effort. The dynamics of effort are illustrated in Figure 1.

Note also that the effort level $e^1_{S,t}$ does not depend on the time when the breakthrough occurred. The reason for it is that the belief about the rival does not change with own breakthrough. Thus, we may compute the path of effort $e^1_{S,t}$ as if the breakthrough happened at time $t = 0$. The firm's effort then follows the path $e^0_{S,t}$ before the breakthrough and switches to the path $e^1_{S,t}$ after the breakthrough, as indicated in Figure 1.

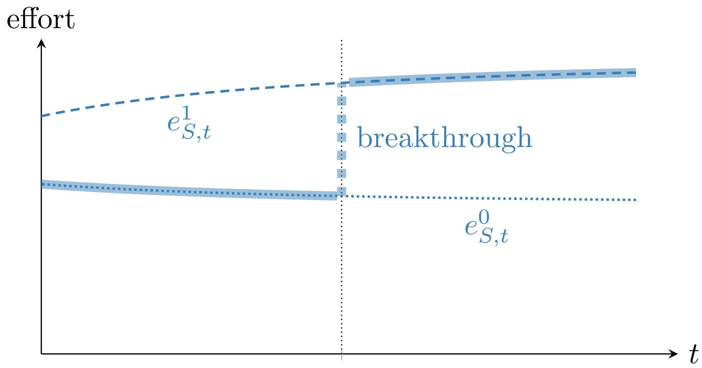


Figure 1: Example of a firm's effort over time in the private information case. The rise occurs as the firm makes the first breakthrough.

6 Patent Race with One Firm Known to Have a Breakthrough

This section explores an asymmetric scenario in which one firm's breakthrough is publicly known, whereas the state of the other firm is its private information. Without loss of generality, assume that the firm known to have a breakthrough is firm A. In other words, firm A is in state 1, and this is common knowledge, whereas firm B's state remains private. We will also say that firm A is uninformed whereas firm B is informed.

We examine which firm, A or B, enjoys strategic advantage. This lays the groundwork for analyzing the game in which firms have the option to reveal their breakthrough. Let the initial belief (i.e., belief at time $t = 0$) that firm B is in state 1 be $\hat{p}$. Due to the memoryless nature of the innovation process, the value functions obtained in this section serve as continuation values in the subgame in which one firm has revealed its breakthrough while the belief about the other firm being in state 1 equals $\hat{p}$.

We analyze *Markov perfect Bayesian equilibria (MPBE)* of the game. Because $p_0^A = 1$, this reduces the need to keep track of the value function $v_t^{0,A}$ and the belief p_t^A. Thus, the original system of six ODEs (see (24)–(26) in Appendix B.1) simplifies to the following system of four ODEs:[15]

$$-\dot{v}_{A,t}^{1,A} = \frac{a}{2}(e_{A,t}^{1,A})^2 - (r + p_{A,t}^B e_{A,t}^{1,B}) v_{A,t}^{1,A}, \tag{7}$$

$$-\dot{v}_{A,t}^{1,B} = \frac{a}{2}(e_{A,t}^{1,B})^2 - (r + e_{A,t}^{1,A}) v_{A,t}^{1,B}, \tag{8}$$

$$-\dot{v}_{A,t}^{0,B} = \frac{a}{2}(e_{A,t}^{0,B})^2 - (r + e_{A,t}^{1,A}) v_{A,t}^{0,B}, \tag{9}$$

$$\dot{p}_{A,t}^B = (1 - p_{A,t}^B)(e_{A,t}^{0,B} - p_{A,t}^B e_{A,t}^{1,B}), \tag{10}$$

along with the identities

$$a e_{A,t}^{1,A} = v - v_{A,t}^{1,A}, \qquad a e_{A,t}^{1,B} = v - v_{A,t}^{1,B}, \qquad a e_{A,t}^{0,B} = v_{A,t}^{1,B} - v_{A,t}^{0,B}, \tag{11}$$

the initial condition $p_{A,0}^B = \hat{p} \in [0,1)$, and the constraints $v_{A,t}^{0,B}, v_{A,t}^{1,A}, v_{A,t}^{1,B} \in (0, v)$, $v_{A,t}^{0,B} < v_{A,t}^{1,B}$, and $p_{A,t}^B \in [0,1)$, for all $t \geq 0$. In addition, to satisfy the Markov condition, efforts must be such that there exist functions $E_A^{k,j}$ such that $e_{A,t}^{k,j} = E_A^{k,j}(p_{A,t}^B)$ for any $(k,j) \in \{(1,A),(1,B),(0,B)\}$ and $t \geq 0$. Equivalently, $v_{A,t}^{k,j}$ must be such that there exist functions $V_A^{k,j}$ such that $v_{A,t}^{k,j} = V_A^{k,j}(p_{A,t}^B)$ for any $(k,j) \in \{(1,A),(1,B),(0,B)\}$ and $t \geq 0$.

Note that from the perspective of the informed firm B, the perceived hazard rate of the rival's patenting becomes $e_{A,t}^{1,A}$. Therefore, it faces the risk-adjusted discount factor $r + e_{A,t}^{1,A}$, which appears in ODEs (8)–(9). On the other hand, for the uninformed firm A, the perceived hazard rate of the rival's patenting still includes the belief $p_{A,t}^B$, so the

[15] We use subscript A to indicate that firm A is the one known to have a breakthrough.

risk-adjusted discount factor in ODE (7) is $r + p^B_{A,t} e^{1,B}_{A,t}$.

Following a similar line of reasoning as in the previous section, we prove the existence of a unique equilibrium in this game and obtain the dynamics of equilibrium efforts. The proofs of Propositions 4 and 5 are provided in Supplementary Appendix E. The analysis parallels that of the symmetric case (Appendix C): we establish existence and uniqueness of a critical point for the four-dimensional ODE system, characterize its stability, and prove convergence. While the structure is analogous, the proofs involve additional technical challenges due to the higher dimensionality of the system.

Proposition 4. *In a patent race in which one firm A's breakthrough is publicly known and the other firm B's state remains private, if the initial belief $\hat{p}$ is strictly below the critical point belief $p^B_{A,*}$, there exists a unique Markov perfect Bayesian equilibrium.*

Proposition 5. *Consider the equilibrium from Proposition 4. Then firm A's effort $e^{1,A}_{A,t}$ increases over time. Moreover, firm B's effort $e^{1,B}_{A,t}$ after achieving a breakthrough increases over time.*[16]

This result is analogous to the one from the symmetric version of the game with both firms in an unknown state. However, in this case the result may appear surprising, because firm B changes its effort over time even though its belief about firm A's state remains fixed. The result is driven by second-order beliefs: firm B knows that firm A increasingly expects B to be in state 1. Consequently, firm A becomes increasingly rivalrous over time, and firm B responds by increasing its effort as well.

In the remainder of this section we explore additional questions arising due to information asymmetry. First, we focus on the case in which both firms have achieved a breakthrough, but only one firm's advancement is publicly known. We examine which firm invests more aggressively in R&D and determine who gains a strategic advantage (Proposition 6). Second, we compare the steady states of the asymmetric case when firm

[16]In addition, it can be shown numerically that, in contrast, firm B's effort $e^{0,B}_{A,t}$ before making the first breakthrough decreases over time, as in the symmetric version of the game. Furthermore, it can be shown numerically that firm B's effort jumps up upon the arrival of its first breakthrough, i.e., $e^{0,B}_{A,t} < e^{1,B}_{A,t}$.

A is known to have a breakthrough and the case with both firms in an unknown state analyzed in Section 5 (Proposition 7).

Proposition 6. *Suppose that both firms A and B have achieved a breakthrough, but only firm A's state is publicly known. The informed firm B exerts higher effort (i.e., $e_{A,t}^{1,A} < e_{A,t}^{1,B}$) but its continuation value is lower than the uninformed firm A's (i.e., $v_{A,t}^{1,A} > v_{A,t}^{1,B}$). However, from the perspective of a fully informed third party (or equivalently, from firm B's informed perspective), the informed firm B is better off than the uninformed firm A.*

The first statement establishes that the uninformed firm A invests less aggressively in R&D due to uncertainty about the rival's state. Although firm A appears to have a higher continuation value, this comparison is potentially misleading because firm A is unaware of firm B's breakthrough. Firm A's continuation value is calculated based on its belief $p_{A,t}^{B}$ rather than the true state. A more accurate comparison is provided by the final statement, which evaluates continuation values from the perspective of a fully informed third party (or equivalently, from firm B's informed perspective).

Although the result may seem intuitive at first glance, it reflects a subtle and nontrivial strategic tension. Possessing additional information appears beneficial, as it provides a firm with the option to act on that knowledge. However, being known to have superior information can place the firm at a strategic disadvantage—prompting rivals to adjust their behavior in ways that undermine the informed firm's position. The trade-off between informational advantage and strategic exposure is therefore more delicate than it initially appears.

Proposition 7. *The following comparison of the critical point effort levels between the symmetric private information case and the case where firm A is known to have a breakthrough holds:*

$$e_{A,*}^{1,A} < e_{S,*}^{1} < e_{A,*}^{1,B} \qquad \text{and} \qquad e_{A,*}^{0,B} < e_{S,*}^{0}.$$

In addition, the beliefs satisfy $p_{A,*}^{B} < p_{S,*}$.

The intuition for this proposition builds on the previously established result that firms become more competitive when the race is tight. Compared to the private information case, a successful firm B exerts higher effort in the long run when firm A's breakthrough is publicly known. By contrast, an unsuccessful firm B is discouraged by firm A's breakthrough and consequently exerts less effort. As a result, in the long run, firm B enters the interim state more slowly and exits it more quickly. This leads to a lower critical point belief that firm B has achieved a breakthrough. Consequently, firm A responds by exerting less effort when the rival is aware of its breakthrough.

7 Patent Race with Optional Revelation

This section addresses our central question: Should a firm reveal its breakthrough? In other words, when a firm achieves its first breakthrough, does it gain an advantage if its rival becomes aware of this progress? Recall that revealing a breakthrough has no direct payoff implications. A firm can benefit from disclosure only if it succeeds in discouraging its rival's R&D effort.

In contrast to the private information game, where firms choose their research efforts silently, the possibility of information revelation introduces additional strategic considerations. By revealing its own breakthrough, a firm influences its rival's incentives to disclose as well. In the resulting subgame—where one firm is known to be successful—revelation discourages an unsuccessful rival but encourages a successful one (as shown by Propositions 5 and 6 in the previous section). Consequently, the firm faces a trade-off that depends on how likely it believes the rival to be unsuccessful, how long that state is expected to persist, and whether the rival is likely to reveal its own breakthrough.

We again focus on symmetric *Markov perfect Bayesian equilibria (MPBE)*. As in the previous sections, the firms can condition their actions on the payoff-relevant state, which (for firm j) consists of the state k_t^j and the belief profile (p_t^A, p_t^B), with both firms starting from the initial state 0, so that the initial belief profile is $(p_0^A, p_0^B) = (0, 0)$.

This restriction allows us to use the solution method of backward induction. However, compared to the previous sections, the action space is extended by the action of *revealing* own breakthrough, once the firm is *successful* (i.e., has reached state 1).[17]

Solving the game using backward induction involves the following four steps.[18] First, suppose that both firms have revealed already. The resulting subgame is then equivalent to the complete information game with both firms being in state 1, as analyzed in Section 3. Therefore, the first step does not require any additional analysis.

Second, suppose both firms are in state 1, but only j has revealed it. We show that its rival never wants to reveal a breakthrough. Consequently, the continuation values in this game coincide with those in the private information game with firm j known to be successful (Section 6).

Third, knowing how firms behave in the described subgames, we study the incentives to reveal before anyone has revealed. We identify two types of symmetric equilibria in pure strategies: a *no-revelation equilibrium*, where firms never reveal their breakthrough and an *instant-revelation equilibrium*, where the first firm to get success reveals it immediately. In addition, we identify a *mixed-revelation equilibrium*, where the firms use mixed strategies and reveal their breakthroughs only with a certain probability.

Fourth, we show that the equilibrium is always unique and is one of the three types described above. In addition, we characterize the parameter values under which each type of equilibrium arises.

[17]Note that off-path beliefs do not play a role here as the only observable action is the verifiable disclosure of a breakthrough. Prior to disclosure, the belief remains strictly less than 1 at any finite time. Thus, non-disclosure is always on the equilibrium path. Disclosure, being verifiable, induces an immediate jump in beliefs to 1.

[18]We would like to point out that, unfortunately, due to the complex structure of equations, the explicit characterization of equilibria is not possible. Therefore, we accompany the implicit characterization by results obtained by the use of numerical methods. Any result that uses numerical methods in its proof is clearly marked as a numerical result. Concretely, thanks to the normalization discussed in footnote 5, the model reduces to a single effective parameter ρ. This allows us to solve the relevant systems of ODEs and verify each claim for every value of ρ on a dense grid. The code is written in Python and is available on author's homepages as well as upon request.

Never Reveal Second

We now proceed with the second step. Consider the subgame in which one firm (say, firm A) has already disclosed its breakthrough. If the rival (firm B) has also achieved a breakthrough, it decides whether to reveal it. Upon revealing, both firms are known to be in state 1, and firm B receives the continuation value v_C^{11} from the complete-information setting. By contrast, if B withholds information about its breakthrough, its continuation value depends on firm A's belief p_t^B and equals $V_A^{1,B}(p_t^B)$, as introduced in Section 6. Comparing these continuation values yields the following proposition.

Proposition 8. *In any equilibrium, a firm never reveals its breakthrough after the rival's revelation.*

The proof begins by establishing that $V_A^{1,B}(p) > v_C^{11}$ for any $p \in [0, p_{A,*}^B]$, which implies that firm B strictly prefers not to reveal its breakthrough. This confirms that non-disclosure is indeed firm B's equilibrium strategy. The second part of the proof shows that this strategy is the only equilibrium strategy. This step is more intricate and involves ruling out the possibility of any alternative strategy that could yield the same or higher payoff under equilibrium conditions.

The intuition behind the proposition has already been illustrated in the complete information case, where firm B's revelation leads firm A to increase its effort (see Proposition 1). A similar, though weaker, effect persists under optional revelation. As revealing a breakthrough has no direct payoff implications, the decision to disclose is guided solely by its strategic effect on the rival's effort. Firm B therefore has a strict incentive to withhold its breakthrough to avoid motivating firm A to intensify its R&D effort.

Proposition 8 implies that once firm A reveals its breakthrough, firm B will never disclose its own. As a result, the game proceeds as in the private information setting with one firm (namely, firm A) known to be successful. Therefore, if firm A reveals its breakthrough before firm B does, it secures the continuation value $V_A^{1,A}(p_t^B)$, whereas firm B receives the continuation value $V_A^{k,B}(p_t^B)$, where k denotes firm B's true state and

p_t^B represents firm A's belief that firm B has also made a breakthrough (see Section 6).

Knowing how the game proceeds after a breakthrough is revealed allows us to evaluate the incentives for a firm to disclose its progress before either firm has revealed. Numerical analysis (see footnote 18) shows that, across all parameter values, the inequalities $E_A^{0,B}(p) < E_S^0(p)$ and $E_A^{1,B}(p) > E_S^1(p)$ hold for all $p \in [0, 1)$. Figure 2 illustrates this comparison: the solid line depicts the rival's effort if firm A does not reveal, whereas the dashed line shows the rival's effort if A does reveal. This indicates that firm A's revelation discourages the rival's effort while the rival remains in state 0. However, once the rival achieves a breakthrough, the knowledge of A's success leads it to exert even greater effort.

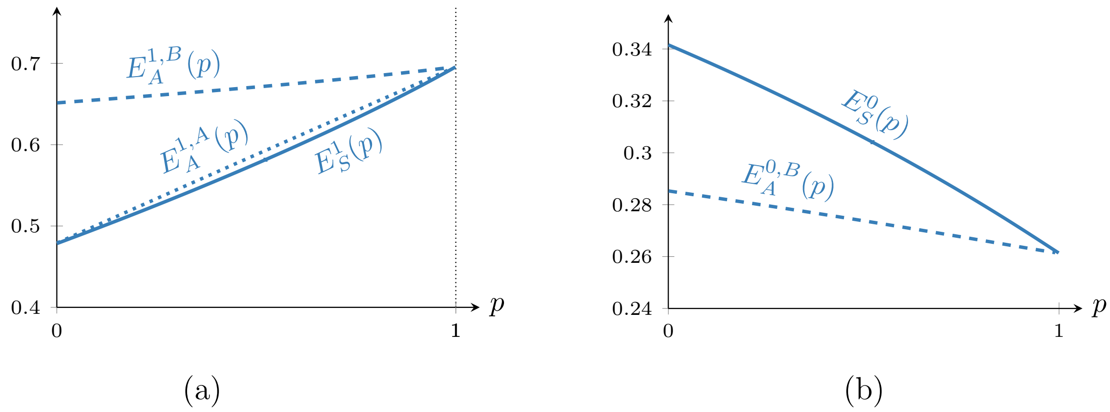


Figure 2: Illustration of the effect of firm A's revelation on the rival's effort (for parameter value $r = 0.1$ and using normalization $a = v = 1$).

Pure-strategy Equilibria

In this subsection, we proceed with the third step. We analyze the two possible pure-strategy equilibria in the patent race with optional revelation: one in which firms never disclose their breakthroughs, and another in which firms reveal their breakthroughs immediately upon achieving them. The first type is the *no-revelation equilibrium*, defined as a symmetric equilibrium in which both firms adopt the strategy of never revealing a breakthrough. In this case, no firm discloses, and the game proceeds as in the symmetric private information setting described in Section 5. Specifically, at time t the posterior belief about either firm having a breakthrough is $p_{S,t}$. The corresponding continuation value for each firm in the no-revelation equilibrium equals $V_S^1(p_{S,t})$ (Section 5).

In order for no revelation to establish an equilibrium, we need to verify that no firm has an incentive to deviate by revealing its breakthrough. Indeed, if one firm (say, firm A) does deviate, the game transitions to the subgame where one firm is known to be successful (Section 6). The continuation value from deviating equals $V_A^{1,A}(p_{S,t})$.[19] Comparing these continuation values yields the following proposition.

Proposition 9. *A no-revelation equilibrium exists if and only if $V_A^{1,A}(p) \leq V_S^1(p)$ for all $p \in [0, p_{S,*})$, where $p_{S,*}$ denotes the critical point belief in the symmetric private information version of the game (see Proposition 3). In this case, the resulting equilibrium path coincides with that of the symmetric private information setting, where firms never disclose their breakthroughs.*

Due to the complexity of the model, we do not have explicit formulas for the value functions from the proposition, making it impossible to analytically verify the inequality for all $p \in [0, p_{S,*})$. Instead, we rely on numerical methods. For each value of the research difficulty $\rho = ar/v$, we can solve for the critical point and compute the corresponding value functions by integrating backward in time.[20] It turns out (see Lemma D.2 in the Appendix) that for a no-revelation equilibrium to exist, it is both necessary and sufficient that the inequality holds at $p = 0$, i.e.,

$$V_A^{1,A}(0) \leq V_S^1(0).$$

Intuitively, if a firm would have no incentives to reveal its breakthrough at time $t = 0$ when still $p_t = 0$, it also has no incentive to do so later. The above inequality yields a simple threshold condition: a no-revelation equilibrium exists if and only if the research difficulty ρ does not exceed a certain threshold ρ_N that we can numerically approximate

[19] By symmetry, a firm j's continuation value depends only on the pair of posterior beliefs, not on whether $j = A$ or $j = B$.

[20] Recall that, as discussed in footnote 5, we can normalize the model to a single parameter $\rho = ar/v$.

as $\rho_N \approx 0.1113$.

The second type is an *instant-revelation equilibrium*: a symmetric equilibrium in which both firms reveal a breakthrough immediately, unless the rival has already revealed. Once a firm reveals, the game proceeds as in the private-information setting with one firm known to have a breakthrough (see Section 6). Until the first revelation, the public state remains the initial state. By the Markov property and symmetry, both firms exert the same constant effort during this phase. Let e_I^0 denote the equilibrium effort and v_I^0 the corresponding continuation value.

An instant-revelation equilibrium exists if no firm has an incentive to delay disclosure after a breakthrough. This requirement yields the following implicit condition for existence.

Proposition 10. *An instant-revelation equilibrium exists if and only if*

$$0 \geq \frac{1}{2a}\big[v - V_A^{1,A}(0)\big]^2 + e_I^0 V_A^{1,B}(0) - (r + e_I^0)V_A^{1,A}(0), \tag{12}$$

where $e_I^0 \in \big(0, V_A^{1,A}(0)/a\big)$ *is the unique positive solution to the quadratic equation*

$$0 = \frac{a}{2}(e_I^0)^2 + e_I^0 V_A^{0,B}(0) - (r + e_I^0)\big[V_A^{1,A}(0) - a e_I^0\big]. \tag{13}$$

As in the earlier analysis, numerical methods confirm that condition (12) holds if and only if the research difficulty $\rho = ar/v$ exceeds a critical threshold, $\rho_I \approx 0.1706$. The logic is intuitive. A no-revelation equilibrium arises when research difficulty ρ is low, whereas an instant-revelation equilibrium prevails when ρ is high. Revealing discourages an unsuccessful rival but encourages a successful one. When research is difficult, the rival is expected to remain unsuccessful for longer, so the discouragement effect dominates, making immediate revelation optimal (provided the rival has not already revealed). Conversely, when research is easy, the encouragement effect dominates, and firms prefer to keep their breakthroughs secret.

Because $\rho_N < \rho_I$, the no-revelation and instant-revelation equilibria do not exist simultaneously for any given ρ. The attractiveness of revealing depends on the rival's strategy and research difficulty: when a rival is expected to reveal, the benefit from revealing diminishes, whereas higher research difficulty increases the value of discouraging an unsuccessful rival. Specifically, for $\rho > \rho_N \approx 0.1113$, a firm would profitably deviate and reveal in the no-revelation equilibrium. Yet for $\rho < \rho_I \approx 0.1706$, a firm would prefer to withhold in the instant-revelation setting. Hence, the two regions do not intersect.

Figure 3 illustrates the incentives to reveal. The solid line plots the difference between the continuation value from revealing and the equilibrium value from not revealing in the no-revelation equilibrium, i.e., $V_A^{1,A}(0) - V_S^1(0)$. The dotted line shows the difference between the equilibrium continuation value in the instant-revelation equilibrium and the value from not revealing in that same equilibrium, i.e., $V_A^{1,A}(0) - \tilde{v}_I^1$.[21]

When both curves lie below zero (i.e., for low ρ), revealing is unprofitable in either equilibrium, and the no-revelation equilibrium prevails. When both curves lie above zero (i.e., for high ρ), revealing dominates, and the instant-revelation equilibrium prevails. In the intermediate range—between the two thresholds—neither pure-strategy equilibrium is sustainable.

These insights motivate the analysis in the next subsection, where we explore the structure of the equilibrium in the intermediate region $\rho \in (\rho_N, \rho_I)$. We show that the equilibrium is unique and it takes the form of a mixed-strategy equilibrium in which firms randomize over the disclosure of their breakthrough.

Mixed-strategy Equilibria

Having examined the two extreme pure-strategy equilibria, we now turn to mixed-strategy equilibria, in which firms may randomize the timing of disclosure. In such equilibria, revelation events are modeled as a Poisson process, with the hazard rate capturing the likelihood of revelation in the interval $[t, t + \Delta t]$. Formally, a mixed strategy for firm

[21] The value $\tilde{v}_I^1$ is formally introduced in the proof of Proposition 10 in the Appendix.

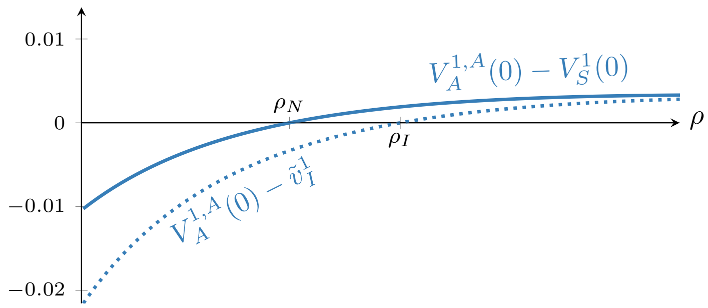


Figure 3: Incentives to reveal as a function of research difficulty ρ. Threshold crossings indicate the equilibrium regions (using normalization $a = v = 1$).

j prior to any revelation consists of a triple of non-negative, right-continuous functions $(e^{1,j}_{M,t}, e^{0,j}_{M,t}, g^j_{M,t})$, defined for $t \geq 0$.[22] Here, $e^{1,j}_{M,t}$ and $e^{0,j}_{M,t}$ denote firm j's efforts in states 1 and 0, respectively, and $g^j_{M,t}$ is the hazard rate at which firm j reveals its success, from the perspective of the rival $-j$.[23] As in Section 4, we define the posterior $p^j_{M,t}$ to be firm $-j$'s belief that firm j has made a breakthrough, conditioned on the fact that firm j has neither patented nor revealed its breakthrough (see also the discussion succeeding (2)). In a symmetric equilibrium, $p^A_{M,t} = p^B_{M,t}$, so we drop the index j and write $p_{M,t}$.

Three features of this definition are noteworthy. First, although the definition specifies the hazard rate at which firm j reveals from its rival's perspective, it does not distinguish between revealing a newly made breakthrough versus one made earlier.[24] Second, although our definition of $g^j_{M,t}$ is unconventional, it captures the essential features of mixed-strategy equilibria. In simple simultaneous-move games, a firm's mixing probability is pinned down by the rival's indifference condition. Analogously here, from the rival's perspective, the indifference condition depends on its belief about the timing of revelation, which aggregates the mixing probability and the likelihood that the rival has already succeeded. Third, this definition encompasses both no-revelation and instant-revelation strategies. When $g^j_{M,t} = 0$, firm j does not reveal a breakthrough at time t.

[22] We use subscript M to denote the mixed-strategy equilibrium.

[23] The rival does not observe firm j's state, so the hazard rate reflects an average over the rival's beliefs across the states and, thus, does not condition on the state.

[24] One interpretation of firm j's strategy is that it reveals a breakthrough upon its arrival with probability $\hat{g}^j_{M,t} \in [0, 1]$. In this case, $g^j_{M,t} = (1 - p^j_{M,t}) e^{0,j}_{M,t} \hat{g}^j_{M,t}$, as from $-j$'s perspective, firm j is in state 0 with probability $1 - p^j_{M,t}$, makes a breakthrough at rate $e^{0,j}_{M,t}$, and reveals it with probability $\hat{g}^j_{M,t}$.

Conversely, when $g^j_{M,t} = e^{0,j}_{M,t}$, the firm reveals with certainty, as the revelation hazard rate equals the breakthrough arrival rate.

The strategy must satisfy the Markov property, which in the case of a symmetric equilibrium means that there exist functions E^0_M, E^1_M and G_M such that

$$e^{1,j}_{M,t} = E^1_M(p_{M,t}), \quad e^{0,j}_{M,t} = E^0_M(p_{M,t}), \quad g^j_{M,t} = G_M(p_{M,t}), \quad \text{for all } t \geq 0.$$

In other words, the strategy depends on time t only through the belief $p_{M,t}$. For regularity, we require the functions E^0_M, E^1_M and G_M to be piecewise continuous. Furthermore, we denote the associated continuation values as $v^{0,j}_{M,t} = V^0_M(p_{M,t})$ and $v^{1,j}_{M,t} = V^1_M(p_{M,t})$.

The belief dynamics in the mixed-strategy case differ from the private information case by accounting for revelation. Firm $-j$'s belief about firm j's breakthrough is now conditioned on both the absence of a second breakthrough and the lack of revelation of the first one. The law of motion must therefore incorporate the revelation hazard rate $g^j_{M,t}$.

Lemma 2. *Whenever the hazard rate $g^j_{M,t}$ is finite, the posterior belief $p^j_{M,t}$, conditional on no revelation by either firm, follows the law of motion $\dot{p}^j_{M,t} = (1-p^j_{M,t})(e^{0,j}_{M,t} - p^j_{M,t}e^{1,j}_{M,t} - g^j_{M,t})$.*

The dynamics of the belief is now governed by the relation between the hazard rates $e^{0,j}_{M,t}$ and $p^j_{M,t}e^{1,j}_{M,t} + g^j_{M,t}$, where $p^j_{M,t}e^{1,j}_{M,t}$ is the hazard rate of firm j patenting from the perspective of firm $-j$. As in the private information case, the former represents the hazard rate of reaching state 1. The latter now includes both the hazard rate of leaving state 1 and the hazard rate of revelation, as firm $-j$ conditions its actions on the fact that firm j has neither patented its second breakthrough nor revealed its first breakthrough.

Next, we focus on symmetric equilibria and simplify notation by dropping the index j. The following lemma establishes that the hazard rate $g_{M,t}$ is finite and characterizes key dynamic properties of the equilibrium.

Lemma 3. *In any symmetric Markov perfect Bayesian equilibrium, the hazard rate* $g_{M,t}$ *is finite and the belief prior to any revelation* $p_{M,t}$ *is non-decreasing over time. Moreover, in a symmetric equilibrium other than instant-revelation, firms always reveal with probability strictly less than 1.*

The law of motion from Lemma 2 together with $p_{M,t}$ being non-decreasing implies that $g_{M,t} \leq e^0_{M,t} - p_{M,t}e^1_{M,t}$, for any $t \geq 0$. This imposes an upper bound on the hazard rate $g_{M,t}$. Note also that it implies that instant-revelation (i.e., $g_{M,t} = e^0_{M,t}$) can occur only when $p_{M,t} = 0$. Conversely, whenever there is uncertainty about the rival's success (i.e., $p_{M,t} > 0$), then a firm delays revelation with a positive probability (i.e., $g_{M,t} < e^0_{M,t}$).

Now consider an equilibrium other than the instant-revelation equilibrium. As in the previous sections, the dynamics when firms use mixed strategies are governed by the following system of ODEs (see Appendix D.4 for details):

$$-\dot{v}^1_{M,t} = \frac{a}{2}(e^1_{M,t})^2 + g_{M,t}V^{1,B}_A(p_{M,t}) - (r + g_{M,t} + p_{M,t}e^1_{M,t})v^1_{M,t}, \tag{14}$$

$$-\dot{v}^0_{M,t} = \frac{a}{2}(e^0_{M,t})^2 + g_{M,t}V^{0,B}_A(p_{M,t}) - (r + g_{M,t} + p_{M,t}e^1_{M,t})v^0_{M,t}, \tag{15}$$

$$\dot{p}_{M,t} = (1 - p_{M,t})(e^0_{M,t} - p_{M,t}e^1_{M,t} - g_{M,t}), \tag{16}$$

together with the identities

$$ae^1_{M,t} = v - v^1_{M,t}, \qquad ae^0_{M,t} = v^1_{M,t} - v^0_{M,t}, \tag{17}$$

the initial condition $p_{M,0} = 0$, and the inequalities $v^0_{M,t}, v^1_{M,t} \in (0, v)$, $v^0_{M,t} < v^1_{M,t}$, $p_{M,t} \in [0,1]$, and $g_{M,t} \in [0, e^0_{M,t} - p_{M,t}e^1_{M,t}]$ for all $t \geq 0$. Compared to the ODEs in the private information case, equations (14)–(15) include an additional term $g_{M,t}V^{k,B}_A(p_{M,t})$, which corresponds to the case where the rival reveals its breakthrough. This occurs with hazard rate $g_{M,t}$, yielding the continuation value $V^{k,B}_A(p_{M,t})$. In addition, the risk-adjusted discount rate now also includes the hazard rate $g_{M,t}$ of the rival revealing its breakthrough.

The above system of ODEs determines the dynamics of the continuation values, efforts, and beliefs for a given trajectory of the hazard rate $g_{M,t}$. Because the firm always has the option to reveal, in equilibrium, the value function after achieving a breakthrough needs to be greater than or equal to the value function from revealing, i.e., $v^1_{M,t} \geq V^{1,A}_A(p_{M,t})$. If this inequality is strict, the firm will not reveal its breakthrough at time t, and thus $g_{M,t} = 0$. Conversely, whenever a firm is mixing at time t (i.e., $g_{M,t} > 0$), its continuation value is the same as after revelation, leading to the indifference condition:

$$v^1_{M,t} = V^{1,A}_A(p_{M,t}), \qquad \text{whenever } g_{M,t} > 0. \tag{18}$$

In equilibrium (other than instant revelation), this indifference condition can be used to determine the hazard rate $g_{M,t}$.

The following proposition provides a detailed characterization of an equilibrium, which we refer to as the *mixed-revelation equilibrium.*[25]

Proposition 11 (partially numerical). *If an equilibrium involves randomization, then there exists some $T > 0$ such that:*

(i) $V^{1,A}_A(p_{M,T}) = V^1_S(p_{M,T})$ *and* $p_{M,T} < p_{S,*}$.

(ii) For $t \in [0, T)$, firms randomize over revelation. The hazard rate $g_{M,t}$ is positive, decreases over time, and satisfies:

$$g_{M,t} = \frac{\frac{a}{2}[E^{1,A}_A(p_{M,t})]^2 - (r + e^0_{M,t})V^{1,A}_A(p_{M,t}) + [e^0_{M,t} - p_{M,t}E^{1,A}_A(p_{M,t})]\overline{V}(p_{M,t})}{\overline{V}(p_{M,t}) - V^{1,B}_A(p_{M,t})}, \tag{19}$$

where $\overline{V}(p_{M,t}) = V^{1,A}_A(p_{M,t}) + (1 - p_{M,t})(V^{1,A}_A)'(p_{M,t})$.

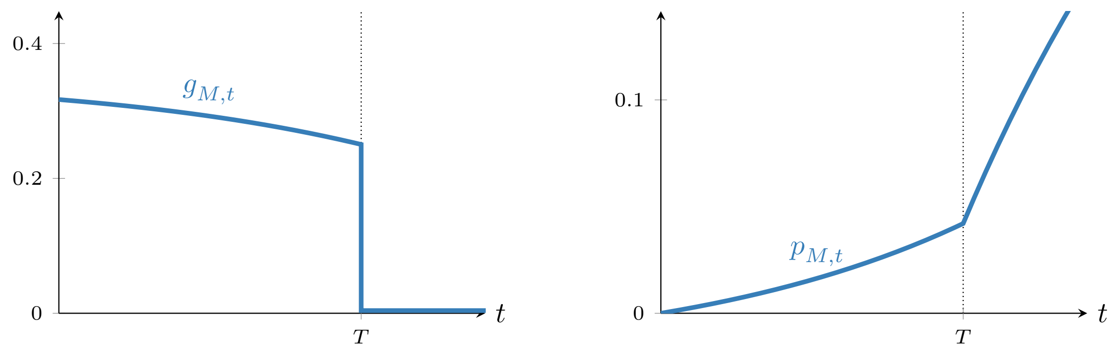
(iii) For $t \geq T$, firms do not reveal their breakthrough and $g_{M,t} = 0$.

The proposition implies that mixing is sustainable only up to a finite time T, after which firms no longer reveal. Equilibrium behavior thus consists of two distinct phases.

[25]The equilibrium is characterized analytically, whereas its uniqueness as a solution is demonstrated numerically.

In the first phase—before time T—firms mix over revelation, with a strictly positive hazard rate. The value function $v^0_{M,t}$ evolves according to the differential equation (15), whereas $v^1_{M,t}$ satisfies the indifference condition (18). During this phase, the belief $p_{M,t}$ increases over time, following the law of motion (16). The mixing phase concludes once the belief reaches the critical value identified in part (i) of the proposition. At this point, the value of revealing becomes equal to the value of withholding, conditional on no future revelations. Consequently, firms cease to reveal, and the game transitions into a second phase, which proceeds as in the private information benchmark. From time T onward, the value functions coincide with those in the private information case, i.e., $v^1_{M,t} = V^1_S(p_{M,t})$, $v^0_{M,t} = V^0_S(p_{M,t})$, and the belief $p_{M,t}$ evolves according to the law of motion (5).

In the proof of Proposition 11, we also establish that firms' hazard rates of revealing are strategic substitutes (see the proof of Lemma D.4). That is, firm j prefers to delay revelation when the rival's hazard rate of revealing exceeds a certain time-dependent threshold, denoted $\bar{g}^{-j}_{M,t}$. Conversely, firm j prefers to reveal immediately when the rival's hazard rate is below this threshold. Therefore, the indifference condition (19), which governs the firm's mixing behavior, can equivalently be viewed as requiring the firm's own hazard rate of revealing to track this threshold whenever it randomizes over disclosure.



(a) (b)

Figure 4: Hazard rate $g_{M,t}$ (left) and belief $p_{M,t}$ (right) over time in the mixed-revelation equilibrium (example for $\rho = 0.15$, the left plot uses the normalization $a = v = 1$).

The characterization of the mixed-revelation equilibrium in Proposition 11 also offers a practical method for computing it. Given the critical point belief $p_{S,*}$ from the symmetric private information case, consider a value $\overline{p}_M \in (0, p_{S,*})$ such that $V_A^{1,A}(\overline{p}_M) = V_S^1(\overline{p}_M)$, and set $V_M^0(\overline{p}_M) = V_S^0(\overline{p}_M)$.[26] To solve for the equilibrium, we compute $V_M^0(p)$ by solving the ODE

$$(V_M^0)'(p) = -\frac{\frac{a}{2}[E_M^0(p)]^2 + G_M(p)V_A^{0,B}(p) - [r + G_M(p) + pE_M^1(p)]V_M^0(p)}{(1-p)\left[E_M^0(p) - pE_M^1(p) - G_M(p)\right]}$$

backward from $\overline{p}_M$ to 0. Note that the ODE features belief p as independent variable instead of time t. This can be obtained by a simple change of variables. Indeed, the numerator of the right-hand side corresponds to the term $-\dot{v}_{M,t}^0$ and the denominator to $\dot{p}_{M,t}$ from equations (15) and (16), respectively. Such a formulation of the ODE is more useful here. Having solved for $V_M^0(p)$ over the interval $[0, \overline{p}_M]$, we then reconstruct the time path of beliefs. Starting from $p_{M,0} = 0$ at time $t = 0$, we simulate forward using the same law of motion (16) until the belief reaches $\overline{p}_M$. The time at which this occurs defines the mixing deadline T, i.e., $p_{M,T} = \overline{p}_M$.

Equilibrium Characterization and Properties

To summarize, we have identified three distinct equilibrium types, each corresponding to a different range of the research difficulty parameter ρ. When research is relatively easy (ρ low), firms never reveal their breakthroughs, resulting in a no-revelation equilibrium. When research is relatively difficult (ρ high), breakthroughs are disclosed immediately, yielding an instant-revelation equilibrium. In the intermediate region of ρ, neither pure-strategy equilibrium is sustainable. Instead, the unique equilibrium takes the form of a mixed-revelation equilibrium, in which firms randomize over disclosure, as characterized in Proposition 11.

Proposition 12 (partially numerical)**.** *The patent race with optional revelation admits a unique symmetric Markov perfect Bayesian equilibrium. The equilibrium type depends on the research difficulty $\rho = ar/v$ as follows:*

[26] See Lemma D.3 in the Appendix, which establishes that such a $\overline{p}_M$ exists and is unique.

(a) No-revelation equilibrium *arises for* $\rho \in [0, \rho_N]$*;*

(b) Mixed-revelation equilibrium *arises for* $\rho \in (\rho_N, \rho_I)$*;*

(c) Instant-revelation equilibrium *arises for* $\rho \in [\rho_I, +\infty)$*,*

where the thresholds are approximately $\rho_N \approx 0.1113$ *and* $\rho_I \approx 0.1706$*.*

The intuition is as follows. Revelation discourages effort by an unsuccessful rival but encourages effort by a successful one. When research difficulty is high ($\rho \geq \rho_I$), it is unlikely that the rival has already succeeded or will succeed soon. In this case, the discouragement effect dominates, and firms prefer to reveal their breakthroughs immediately. When research is easy ($\rho \leq \rho_N$), the rival is likely to have already caught up or will do so shortly. Here, the encouragement effect dominates, and firms prefer to conceal their breakthroughs. In the intermediate region (when $\rho_N < \rho < \rho_I$), neither effect dominates. Firms therefore mix over disclosure, balancing the incentive to deter the rival against the strategic benefit of waiting for the rival to reveal first.

Additional insights can be obtained by numerically simulating the mixed-revelation equilibrium for varying values of $\rho \in (\rho_N, \rho_I)$. We observe that the deadline T (the time at which firms stop revealing) increases with the research difficulty ρ, as illustrated in Figure 5(a). As ρ approaches the upper threshold ρ_I, the revelation hazard rate $g_{M,t}$ converges to the innovation rate $e^0_{M,t}$, causing the posterior belief $p_{M,t}$ to grow more slowly. At $\rho = \rho_I$, the posterior remains constant at zero ($p_{M,t} = 0$ for all $t \geq 0$), and the equilibrium type switches to the instant-revelation equilibrium. Conversely, as ρ approaches the lower threshold ρ_N, the deadline T converges to zero, and the equilibrium switches to the no-revelation regime.

Based on the equilibrium deadline T and the corresponding hazard rate of revealing, we can use numerical simulations to compute the probability that one of the firms reveals its first breakthrough before the second breakthrough is completed. This probability depends on the type of equilibrium: (i) it is zero in the no-revelation equilibrium (for $\rho \leq \rho_N$), where neither firm ever discloses a breakthrough; (ii) it equals one in the instant-revelation equilibrium (for $\rho \geq \rho_I$), where both firms reveal their breakthroughs immediately upon discovery; (iii) in the mixed-revelation equilibrium (for $\rho \in (\rho_N, \rho_I)$), the probability is given by

$$\int_0^T 2g_{M,t} \exp\left[-\int_0^t 2\left(p_{M,s}e^1_{M,s} + g_{M,s}\right) ds\right] dt,$$

where $g_{M,t}$ denotes the hazard rate of revelation, and $p_{M,t}e^1_{M,t}$ captures the hazard rate of completing the second breakthrough.[27] For a given value of the research difficulty ρ, this expression can be evaluated numerically. The resulting probability is strictly increasing in ρ, approaching zero as $\rho \searrow \rho_N$ and one as $\rho \nearrow \rho_I$, as shown in Figure 5 (b). This pattern reflects the intuition that more difficult research makes intermediate revelation more likely.

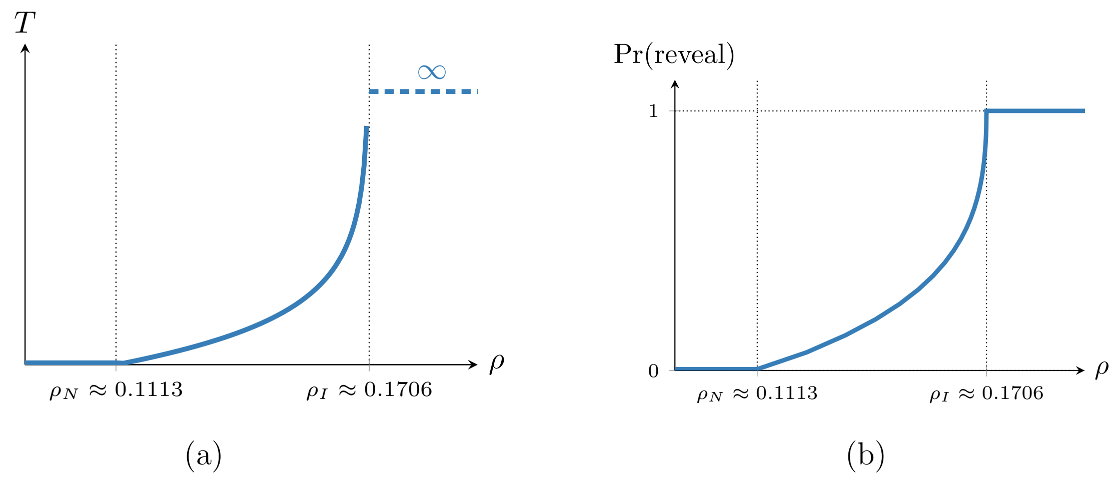


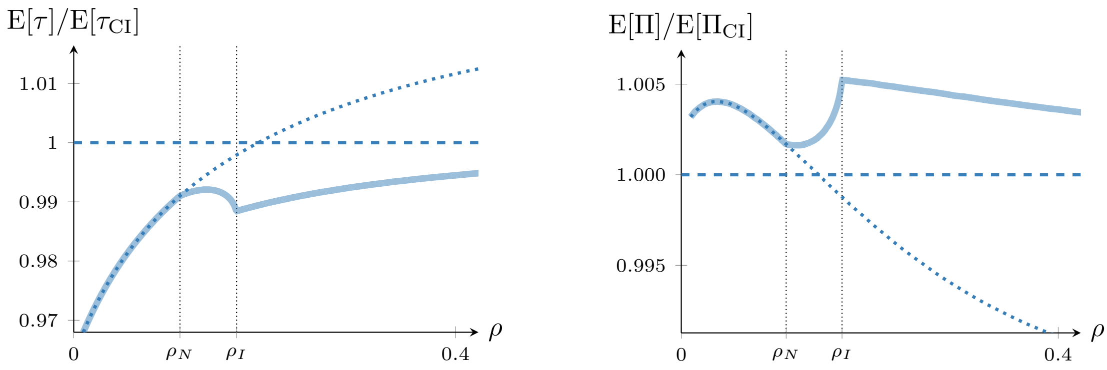
Figure 5: Equilibrium deadline T (left) and the probability that either firm reveals before a patent is made (right) as a function of research difficulty ρ.

We also evaluate welfare using two criteria: expected time to patent and the firms' joint expected profits. Formally, let $\mathrm{E}[\tau]$ denote the expected time until the first firm secures a patent, and let $\mathrm{E}[\Pi] = \mathrm{E}[\Pi^A] + \mathrm{E}[\Pi^B]$ denote the firms' joint expected profits. The expected time to patent captures the social benefit of faster innovation. Because consumers benefit from earlier availability of new technology, shorter expected patenting

[27]The integrand combines the instantaneous revelation hazard rate ($2g_{M,t}$) with the probability that neither firm has patented or revealed by time t.

time serves as a standard proxy for consumer welfare (see, e.g., Loury, 1979).

Figure 6 presents both criteria across the three information settings, expressed relative to the complete information benchmark. In both panels, optional revelation performs weakly better than the complete information as well as the private information setting. The two criteria are closely aligned: the information setting that yields the highest joint profits also produces the shortest expected patenting time. This connection arises because the expected time to patent is driven primarily by effort in state 0, before a firm achieves its first breakthrough. In this state, a firm's willingness to invest depends on the anticipated efficiency of the subsequent race, that is, on how much of the effort after the first breakthrough will be duplicated by a rival in the same position. Under complete information, when both firms are in state 1, the resulting head-to-head competition generates substantial duplication of effort, which depresses the expected value of reaching state 1 and thereby discourages effort in state 0. Information settings that reduce this duplication raise the value of a breakthrough, increase first-stage effort, and accelerate the race. From a policy perspective, a regulator aiming to accelerate innovation should permit but not mandate disclosure. Optional revelation allows firms to self-select: they reveal when the discouragement effect is strong enough to reduce duplication, and conceal otherwise.



(a) (b)

Figure 6: Expected time to patent (left) and joint expected profits (right) as a function of research difficulty ρ, expressed relative to the complete information benchmark (= 1). Optional revelation (solid), private information (dotted), and complete information (dashed) (using normalization $a = v = 1$).

As a final result, we compare the firms' asymptotic behavior under optional revelation to that in the private information case. Recall that once a breakthrough is revealed, the game transitions to the asymmetric case analyzed in Section 6, so the comparison is governed by Proposition 7. Specifically, a firm exerts lower effort following its own revelation than it would under private information. The impact of revelation on the rival depends on its stage in the race: a rival that has not yet achieved a breakthrough exerts less effort compared to the private information case, whereas a rival that has made a breakthrough exerts greater effort as a result of the revelation.

8 Conclusion

This article studies how private information regarding firms' progress shapes strategic behavior in a patent race. We have two objectives: first, to characterize the dynamics when firms do not observe one another's progress toward securing a patent, and second, to identify the conditions under which a firm has an incentive to reveal a breakthrough. We find that a firm's success discourages the effort of an unsuccessful rival while encouraging the effort of a successful rival. Consequently, a firm reveals its success only when it expects its rival to remain behind. We demonstrate that in a patent race with optional disclosure, the equilibrium is unique. Depending on the underlying parameters of the model, we identify three possible equilibrium types. To determine which of these types exists for various parameter configurations, we employ numerical methods.

One potential simplification involves discretizing time over a limited number of periods, which reduces the system of ODEs to a set of multivariate cubic equations. However, even with the minimum number of periods required for disclosure to remain relevant, the model remains analytically intractable. Furthermore, such a model suffers from finite-horizon effects, and the resulting system cannot be solved recursively, as continuation values are pinned down by the terminal boundary conditions whereas the posterior belief is determined in the first period.

We now discuss several potential extensions of our model. Although we have obtained preliminary results for these cases, a full treatment remains beyond the scope of the current manuscript. First, by relaxing the symmetry assumption, one may investigate firms with different R&D efficiencies. Preliminary analysis suggests that if the efficiency gap is sufficiently large, the more efficient firm reveals its progress whereas the less efficient one never discloses. Second, we consider a more general specification of the firms' cost functions. For instance, assuming a power function of the form $c(e) = e^{\gamma}/\gamma$ (where $\gamma > 1$ is a parameter), our qualitative results remain robust; the specific thresholds for each equilibrium type shift based on the curvature of the cost function. Third, allowing for correlated breakthroughs is a natural extension. We conclude that positive correlation would strengthen the encouragement effect of a rival's success, potentially expanding the parameter region in which firms prefer not to reveal.

Finally, we acknowledge that in practice, additional forces may shape disclosure decisions. For instance, revealing a breakthrough may facilitate financing, but it may also generate knowledge spillovers or signal to competitors that a technological hurdle has been overcome. In this article, we abstract from these forces to isolate the strategic role of private information and its revelation in discouraging a rival's R&D effort. These additional considerations represent natural avenues for future research.

A Appendix: Proofs for Section 3 (Benchmark: Patent Race with Observable Progress)

A.1 Derivation of the Optimal Effort (1)

The firm's continuation value in state (k, l) can be characterized recursively as

$$v_C^{k,l} = \max_{e \geq 0} \Big\{ v_C^{k+1,l} e\Delta t - \frac{a}{2} e^2 \Delta t + v_C^{k,l+1} e_C^{l,k} \Delta t \\ + [1 - (e + e_C^{l,k})\Delta t](1 - r\Delta t) v_C^{k,l} + o(\Delta t) \Big\} . \tag{20}$$

The first, third, and fourth term represent the continuation values from making a breakthrough, rival making a breakthrough, and nobody making a breakthrough, respectively, within the time interval $[t, t + \Delta t]$, multiplied by the corresponding probabilities. The second term is the cost of effort e.

Subtracting $v_C^{k,l}$ from both sides of the equation, dividing by $\Delta t > 0$, and taking the limit $\Delta t \searrow 0$, we obtain

$$0 = \max_{e \geq 0} \Big\{ v_C^{k+1,l} e - \tfrac{a}{2} e^2 + v_C^{k,l+1} e_C^{l,k} - (e + e_C^{l,k} + r) v_C^{k,l} \Big\} \tag{21}$$

The expression on the right-hand side is quadratic and concave in effort e. The first order condition for optimal effort then yields $a e_C^{k,l} = v_C^{k+1,l} - v_C^{k,l}$, which is indeed equation (1).

A.2 Proof of Proposition 1

Throughout this section we use the normalization $v = 1$ and $a = 1$, which is without loss of generality (see Lemma B.1 for details).

Substituting the optimal effort $e_C^{k,l} = v_C^{k+1,l} - v_C^{k,l}$ into (21), we obtain the system of equations

$$0 = \tfrac{1}{2}(v_C^{k+1,l} - v_C^{k,l})^2 + e_C^{l,k}(v_C^{k,l+1} - v_C^{k,l}) - r v_C^{k,l}, \qquad k, l \in \{0, 1\} \tag{22}$$

along with the boundary conditions $v_C^{2,l} = 1$ and $v_C^{k,2} = 0$. Moreover, $e_C^{l,k} = v_C^{l+1,k} - v_C^{l,k} > 0$ represents the effort of the rival. Note that this system contains four equations with four unknowns: v_C^{00}, v_C^{10}, v_C^{01}, and v_C^{11}.

The proof of Proposition 1 is based on three lemmas whose proofs are provided in Supplementary Appendix A. Lemma A.1 establishes monotonicity of values in the state. Lemma A.2 establishes uniqueness of the solution. Lemma A.3 provides inequalities and bounds that yield Proposition 1. We first state the lemmas and then prove Proposition 1.

Lemma A.1. *The inequalities* $v_C^{k,l+1} < v_C^{k,l} < v_C^{k+1,l}$ *hold for any* $k, l \in \{0, 1\}$.

Lemma A.2. *The system of four equations* (22) *has a unique solution.*

Lemma A.3. *The following inequalities hold:*

(i) $\underline{E}^{11} := 1 - \dfrac{1}{3+2r} < e_C^{11} < 1 - \dfrac{1}{4+2r} =: \overline{E}^{11}$;

(ii) $e_C^{01} < 1 - e_C^{11} < \dfrac{1}{3+2r} =: \overline{E}^{01}$*;*

(iii) $e_C^{10} < 1 - \dfrac{1}{2+2r} =: \overline{E}^{10}$*;*

(iv) $\underline{E}^{00} := \dfrac{1}{3+2r} < e_C^{00}$.

Proof of Proposition 1. To begin with, by Lemma A.2 the system of equations (22) has a unique solution, which allows us to analyze it. The inequality $e_C^{10} < e_C^{11}$ follows from the fact that $e_C^{1,l} = v_C^{2,l} - v_C^{1,l} = 1 - v_C^{1,l}$, for $l \in \{0, 1\}$, and the inequality $v_C^{11} < v_C^{10}$ (Lemma A.1). The inequality $e_C^{01} < e_C^{00}$ follows from the estimates in Lemma A.3 as $e_C^{01} < \overline{E}^{01} = \underline{E}^{00} < e_C^{00}$. □

B Appendix: Proofs for Section 4 (Patent Race with Unobservable Progress)

The proofs of the lemmas stated in this section are provided in Supplementary Appendix C.

B.1 Optimal Effort and Value Functions

The continuation value of firm j in state k at time t is

$$v_t^{k,j} = \max_{e \geq 0} \left\{ v_t^{k+1,j} e\Delta t - \tfrac{1}{2} a e^2 \Delta t + [1 - (e + p_t^{-j} e_t^{1,-j})\Delta t](1 - r\Delta t) v_{t+\Delta t}^{k,j} + o(\Delta t) \right\},$$

The first term represents the expected value from making a breakthrough within the time interval $[t, t+\Delta t]$, weighted by the probability of it happening. The second term captures the cost of effort e. The third term gives the continuation value when the state (from firm j's perspective) remains unchanged within $[t, t + \Delta t]$, weighted by the corresponding probability. This probability incorporates the hazard rate $p_t^{-j} e_t^{1,-j}$ at which the rival firm patents at time t.

Subtracting $v_{t+\Delta t}^{k,j}$ from both sides of the equation, dividing by $\Delta t > 0$, and taking the limit $\Delta t \searrow 0$, we obtain

$$-\dot{v}_t^{k,j} = \max_{e \geq 0} \left\{ (v_t^{k+1,j} - v_t^{k,j}) e - \tfrac{1}{2} a e^2 - (r + p_t^{-j} e_t^{1,-j}) v_t^{k,j} \right\}.$$

The first-order condition for e implies

$$a e_t^{k,j} = v_t^{k+1,j} - v_t^{k,j}. \tag{23}$$

That is, firm j's optimal effort is proportional to the potential gain from instant completion of the current stage of R&D. This result is analogous to condition (1) derived in the complete information case.

Consequently, for each state $k \in \{0, 1\}$ and firm $j \in \{A, B\}$, we obtain the following differential equation for the value function:

$$-\dot{v}_t^{k,j} = \tfrac{1}{2} a (e_t^{k,j})^2 - (r + p_t^{-j} e_t^{1,-j}) v_t^{k,j}.$$

The game can be summarized by the following system of six ODEs (three for each firm

$j \in \{A, B\}$):

$$-\dot{v}_t^{1,j} = \tfrac{1}{2}a(e_t^{1,j})^2 - (r + p_t^{-j} e_t^{1,-j}) v_t^{1,j}, \tag{24}$$

$$-\dot{v}_t^{0,j} = \tfrac{1}{2}a(e_t^{0,j})^2 - (r + p_t^{-j} e_t^{1,-j}) v_t^{0,j}, \tag{25}$$

$$\dot{p}_t^j = (1 - p_t^j)(e_t^{0,j} - p_t^j e_t^{1,j}), \tag{26}$$

together with conditions (23).

B.2 Normalization of Parameters

The patent race model involves three key parameters: the patent value v, the effort cost multiplier a, and the discount rate r. Without loss of generality, we can normalize $v = 1$ and $a = 1$. This normalization is achieved by choosing appropriate units of value and time. This simplification streamlines the notation in our proofs and reduces numerical analysis to a one-dimensional parameter space. The following lemma formalizes this normalization.

Lemma B.1. *Any equilibrium in the patent race with private information corresponds to an equilibrium of the game with $\hat{v} = 1$, $\hat{a} = 1$, and $\rho = \dfrac{ar}{v}$.*

Throughout the rest of the Appendix we use the normalization $v = 1$ and $a = 1$.

C Appendix: Proofs for Section 5 (Patent Race with Both Firms in Unknown State)

In this section we study symmetric equilibria, therefore we omit $j \in \{A, B\}$ from the superscript. The proofs of the lemmas stated in this section are provided in Supplementary Appendix D.

Using the identities $e_{S,t}^1 = 1 - v_{S,t}^1$ and $e_{S,t}^0 = v_{S,t}^1 - v_{S,t}^0$, the system of ODEs (3)–(5) can be expressed in terms of optimal effort levels and belief as the following problem:

Problem 1. A trajectory $(e^1_{S,t}, e^0_{S,t}, p_{S,t})$ satisfies the following system of ODEs:

$$\dot{e}^1_{S,t} = \tfrac{1}{2}(e^1_{S,t})^2 - (r + p_{S,t}e^1_{S,t})(1 - e^1_{S,t}) \qquad =: F^1_S(e^1_{S,t}, e^0_{S,t}, p_{S,t}) \tag{27}$$

$$\dot{e}^0_{S,t} = \tfrac{1}{2}(e^0_{S,t})^2 - \tfrac{1}{2}(e^1_{S,t})^2 + (r + p_{S,t}e^1_{S,t})e^0_{S,t} \qquad =: F^0_S(e^1_{S,t}, e^0_{S,t}, p_{S,t}) \tag{28}$$

$$\dot{p}_{S,t} = (1 - p_{S,t})(e^0_{S,t} - p_{S,t}e^1_{S,t}) \qquad =: F^p_S(e^1_{S,t}, e^0_{S,t}, p_{S,t}), \tag{29}$$

with the initial condition $p_{S,0} = \hat{p} \in [0, 1)$, constraints $e^1_{S,t} \in (0, 1)$, $e^0_{S,t} \in (0, 1 - e^1_{S,t})$, and $p_{S,t} \in [0, 1)$ for all $t \geq 0$.[28] Assume strategies are Markov in the posterior belief p: there exist functions E^1_S and E^0_S such that $e^1_{S,t} = E^1_S(p_{S,t})$ and $e^0_{S,t} = E^0_S(p_{S,t})$ for all $t \geq 0$.

Although we only need to study the problem with the initial condition containing $\hat{p} = 0$, it is necessary to study the solution with the initial condition $\hat{p} > 0$ to be able to prove the existence of unique solution for $\hat{p} = 0$. Note that the vector (F^1_S, F^0_S, F^p_S) is a continuous function of the vector $(e^1_{S,t}, e^0_{S,t}, p_{S,t})$, and thus the solution vector is an analytic function of time.

The proofs of Propositions 2 and 3 rely on Lemmas C.1–C.6 below. We first establish local existence and uniqueness of a solution in a neighborhood of the critical point (Lemmas C.1–C.3). We then extend this solution uniquely backward in time until the posterior belief reaches 0 (Lemma C.4). Following that, we show that Problem 1 with $\hat{p} = 0$ has a unique solution. We will do this by showing that the solution is unique in the limit as $t \to \infty$, and then that the solution can be extended backward in time to the initial condition $p_{S,0} = 0$. For that purpose we establish Lemmas C.5 and C.6.

Lemma C.1. *The system of ODEs* (27)–(29) *has a unique critical point* $(e^1_{S,*}, e^0_{S,*}, p_{S,*})$ *such that* $e^1_{S,*} \in (0, 1)$, $e^0_{S,*} \in (0, e^1_{S,*})$, *and* $p_{S,*} \in (0, 1)$.

Lemma C.2. *The Jacobian at the critical point of the system of ODEs* (27)–(29) *has one negative eigenvalue and two eigenvalues with a positive real part.*

[28] The constraint $e^0_{S,t} \in (0, 1 - e^1_{S,t})$ follows from the fact that $1 - e^1_{S,t} - e^0_{S,t} = v^0_{S,t} > 0$.

Lemma C.3. *Any solution of Problem 1 converges to the critical point* $(e^1_{S,*}, e^0_{S,*}, p_{S,*})$ *as* $t \to \infty$*. Moreover, if* $p_{S,0} < p_{S,*}$*, then the belief* $p_{S,t}$ *is increasing over time.*

Lemma C.4. *Let* $(e^1_{S,t}, e^0_{S,t}, p_{S,t})$ *be a solution of Problem 1 for* $t > 0$*. Then the constraints of Problem 1 and* $\dot{p}_{S,0} > 0$ *also hold at* $t = 0$*.*

Lemma C.5. *The eigenvector* $\mu_S = (\mu^1_S, \mu^0_S, \mu^p_S)$ *of the Jacobian matrix* J *associated with the negative eigenvalue* λ_1 *satisfies* $\mu^1_S/\mu^p_S > 0$ *and* $\mu^0_S/\mu^p_S < 0$*.*

Lemma C.6. *Let* $(e^1_{S,t}, e^0_{S,t}, p_{S,t})$ *be a solution of Problem 1 with* $\hat{p} \in [0, p_{S,*})$*. If* $\dot{p}_{S,t} > 0$ *for all* $t \geq 0$*, then* $\dot{e}^1_{S,t} > 0$ *and* $\dot{e}^0_{S,t} < 0$ *for all* $t \geq 0$*.*

Proof of Proposition 2. Let p_L be the infimum of the $\mathcal{P}$ values of $\hat{p} \in [0, p_{S,*})$ such that there exists a unique solution $(e^1_{S,t}, e^0_{S,t}, p_{S,t})$ to Problem 1 with initial condition $p_{S,0} = \hat{p}$. Since the system of ODEs (27)–(29) is autonomous, solutions to Problem 1 with progressively decreasing initial condition $\hat{p}$ are extensions of each other except for that the time shifts, and so $\mathcal{P}$ is an interval, and we need to prove that $p_L = 0$. We will do this by contradiction. Suppose that $p_L > 0$.

We begin by showing local existence of unique solution. At the critical point $(e^1_{S,*}, e^0_{S,*}, p_{S,*})$, the Jacobian has one negative eigenvalue and two eigenvalues with a positive real part (Lemma C.2); accordingly, there is a unique stable direction $(\mu^1_S, \mu^0_S, \mu^p_S)$. Moreover, $\mu^p_S \neq 0$ (Lemma C.5). The Hartman–Grobman theorem (Teschl, 2012, Theorem 9.9) therefore guarantees the existence of a locally unique trajectory $(e^1_{S,t}, e^0_{S,t}, p_{S,t})$ of (27)–(29) with $\dot{p}_{S,t} > 0$ for all $t > 0$ that converges to the critical point as $t \to \infty$. Moreover, since $e^1_{S,*} \in (0, 1)$, $e^0_{S,*} \in (0, e^1_{S,*})$, and $p_{S,*} \in (0, 1)$, the constraints on the trajectory are satisfied for all sufficiently large t. The Markov property is also trivially satisfied, as $p_{S,t}$ is strictly increasing. Taking into account that every solution to Problem 1 converges to the critical point (Lemma C.3), there must exist a unique solution to Problem 1 for some $\hat{p} \in [0, p_{S,*})$. We therefore conclude that $p_L < p_{S,*}$.

Let $(\overline{e}_S^1, \overline{e}_S^0, \overline{p}_S)$ be the limit of $(e_{S,0}^1, e_{S,0}^0, p_{S,0})$ corresponding to the solution of Problem 1 with the initial condition $\hat{p}$ as $\hat{p} \searrow p_L$. The limit is well defined since the solutions to Problem 1 with decreasing $\hat{p}$ are embedded in each other.

The vector function (F_S^1, F_S^0, F_S^p) is locally Lipschitz continuous (it is a cubic multivariable polynomial), so by the Picard–Lindelöf theorem (see, e.g., Teschl, 2012, Theorem 2.2), there exists a unique solution $(e_{S,t}^1, e_{S,t}^0, p_{S,t})$ to the system of ODEs (27)–(29) with the initial conditions $(e_{S,0}^1, e_{S,0}^0, p_{S,0}) = (\overline{e}_S^1, \overline{e}_S^0, \overline{p}_S)$ for t on some neighborhood of 0. In fact, the system has a unique solution for all $t \geq 0$, as those correspond to the solution of Problem 1 with initial condition $\hat{p} > p_L$. What is more, the existence of a unique solution for $t < 0$ on some neighborhood of 0 means that by shifting time we can conclude that Problem 1 has a unique solution for some $\hat{p} < p_L$ so long as we ensure that Problem 1 constraints are satisfied. By Lemma C.4, the constraints indeed hold at $t = 0$, including $\dot{p}_{S,0} > 0$. Since the solution of the system of ODEs (27)–(29) is an analytic function, the constraints are also satisfied on some neighborhood of 0. As $\dot{p}_{S,0} > 0$, $p_{S,t} < p_L$ for $t < 0$. We conclude that there exists some $\hat{p} < p_L$ such that Problem 1 with initial condition $p_{S,0} = \hat{p}$ has a unique solution, contradicting $p_L > 0$. □

Proof of Proposition 3. Proposition 2 ensures that a unique solution to Problem 1 with the initial condition $p_{S,0} = 0$ exists. Lemma C.3 establishes that $p_{S,t}$ is strictly increasing. By Lemma C.6, it follows that $e_{S,t}^1$ is strictly increasing and $e_{S,t}^0$ is strictly decreasing over time.

It remains to show that $e_{S,t}^0 < e_{S,t}^1$ for all $t \geq 0$. This is however apparent from equation (28) combined with the fact that $\dot{e}_{S,t}^0 < 0$ (Lemma C.6), which implies that

$$0 > \dot{e}_{S,t}^0 = \tfrac{1}{2}(e_{S,t}^0)^2 - \tfrac{1}{2}(e_{S,t}^1)^2 + (r + p_{S,t}e_{S,t}^1)e_{S,t}^0 > \tfrac{1}{2}(e_{S,t}^0)^2 - \tfrac{1}{2}(e_{S,t}^1)^2.$$

Since both efforts are positive, this yields $e_{S,t}^0 < e_{S,t}^1$. □

D Appendix: Proofs for Section 7 (Patent Race with Optional Revelation)

The proofs of the lemmas stated in this section are provided in Supplementary Appendix F.

D.1 Proof of Proposition 8 (Never Reveal Second)

Proof of Proposition 8. Suppose one firm has revealed a breakthrough already; without loss of generality, let it be firm A. First, the strategy to never reveal is an equilibrium strategy of firm B. Indeed, firm B's continuation value implied by the strategy to never reveal is $V_A^{1,B}(p_{A,t}^B)$, while its continuation value of revealing is $v_C^{11} = V_A^{1,A}(1)$. Applying Lemma E.4, we conclude that $E_A^{1,B}(p_{A,t}^B) < \phi^{1,B}(\sup_{s \geq t} p_{A,s}^B) < \phi^{1,B}(1) = e_C^{1,1}$, and so $V_A^{1,B}(p_{A,t}^B) > V_A^{1,A}(1)$. Thus, a firm has indeed no incentive to reveal.

To show that not revealing second is the only equilibrium, we need to consider any strategy of firm B over revealing second, because firm B's strategy over revelation impacts its rival's effort, and so it impacts its own incentive to reveal. The efforts and continuation values of the two firms follow the same differential equations as those in private information game with one firm being known to be successful, except that the dynamics of $p_{A,t}^B$ is influenced by firm B's strategy over revelation. If firm B is expected to reveal with a positive probability once being successful, then its rival's posterior belief $p_{A,t}^B$ grows slower (or even falls) than it would in the game without revelation; and in the event of firm B revealing, the belief jumps to 1 and stays there. We do not need to describe the exact process of $p_{A,t}^B$, what is relevant is that $p_{A,t}^B$ is less than 1 with a positive probability for a while. We can follow the reasoning from the proof of Lemma E.4 and generalize its results for a stochastic process $p_{A,t}^B$, and obtain the estimate that firm B's continuation value while being in state 1 is strictly more than $1 - \phi^{1,B}(1)$, which is the continuation value it would get after revealing.

In conclusion, regardless of firm B's strategy regarding revealing second, it has a strict incentive not to reveal. Thus, the only equilibrium strategy that firm B can have is not to reveal second. □

D.2 Pure-strategy Equilibria

Proof of Proposition 9. First, the condition $V_A^{1,A}(p) \leq V_S^1(p)$ for all $p \in [0, p_{S,*})$ is necessary for a no-revelation equilibrium to exist. Indeed, suppose that a no-revelation equilibrium exists and that $V_A^{1,A}(p) > V_S^1(p)$ for some $p \in [0, p_{S,*})$. Then firm A has a strict incentive to reveal arrival of a breakthrough at time t such that $p_t = p$, which is a contradiction. Assume that $V_A^{1,A}(p) \leq V_S^1(p)$ for all $p \in [0, p_{S,*})$, and suppose that both firms have the strategy to never reveal. We check that none of the firms has an incentive to deviate. Given that firms do not reveal, their efforts and continuation values are identical to those from the private information version of the game (without revelation). In particular, the continuation value of a successful firm at time t is $V_S^1(p_t)$. In contrast, if a successful firm deviated and revealed, its continuation value would be $V_A^{1,A}(p_t)$, which is no more than $V_S^1(p_t)$ by the assumption.

The no-revelation equilibrium is unique, as the effort levels have to correspond to the unique solution of the private information version of the game (Proposition 2). □

Lemmas D.1 and D.2 below provide some characterization when the condition from Proposition 9 is satisfied.

Lemma D.1 (numerical)**.** $V_A^{1,A}(p_{S,*}) < V_S^1(p_{S,*})$ *for* $\rho < \rho_I$*.*

Note that the inequality is relevant only for $\rho < \rho_I$, since for $\rho \geq \rho_I$ instant revelation is the only equilibrium. We verify the lemma numerically for a range of $\rho \in [0, 0.3]$, as Figure 7 illustrates.

Lemma D.2 (partially numerical)**.** *The inequality* $V_A^{1,A}(0) \leq V_S^1(0)$ *holds if and only if* $\rho = ar/v \leq \rho_N$*, where* $\rho_N \approx 0.1113$*. Moreover, if this inequality holds at* $p = 0$*, then it holds for all* $p \in [0, p_{S,*})$*, i.e.,* $V_A^{1,A}(p) \leq V_S^1(p)$ *for all* $p \in [0, p_{S,*})$*.*

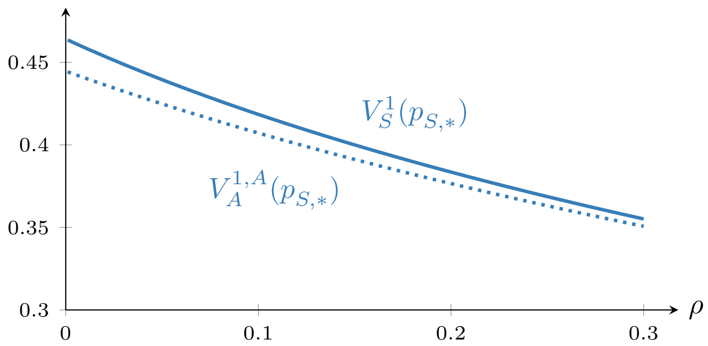


Figure 7: Numerical verification of Lemma D.1 for $\rho \in [0, 0.3]$.

By Proposition 9, a no-revelation equilibrium exists if and only if $V_A^{1,A}(p) \leq V_S^1(p)$ for all $p \in [0, p_{S,*})$. The lemma shows that it suffices to verify this inequality at $p = 0$ alone, rather than on the entire interval, and provides a precise characterization in terms of model parameters: the no-revelation (never-reveal) equilibrium exists if and only if $\rho \leq \rho_N \approx 0.1113$.

The validity of this lemma can be tested numerically. As noted in the main text, due to the complexity of the problem it is not possible to find an explicit solution of the system of ODEs. Proceeding backwards in time from the critical point, we can solve the system numerically. The statements are then obtained by comparing the value functions for different values of the research difficulty ρ.

Intuitively, the lemma holds for the following reason: The functions $V_A^{1,A}(p)$ and $V_S^1(p)$ attain the same value (namely v_C^{11}) for $p = 1$. However, the function $V_S^1(p)$ has a higher curvature, as it corresponds to the value function based on the posterior of both firms changing simultaneously, while the function $V_A^{1,A}(p)$ only reflects that the posterior about firm B changes with p.

Proof of Proposition 10. Suppose that both firms have the strategy to reveal the arrival of a breakthrough instantly (unless the rival has revealed already), and none has done so by time $t \geq 0$. Until either of the firms reveals, the game is static in the sense that each firm is certain that its rival is unsuccessful ($p_t = 0$). As argued in the main text, the effort as well as the value function are constant over time. Moreover, as follows from

(20),

$$v_I^0 = \max_{e\geq 0}\Big\{V_A^{1,A}(0)e\Delta t - \tfrac{1}{2}e^2\Delta t + V_A^{0,B}(0)e_I^0\Delta t + \big[1-(e+e_I^0)\Delta t\big](1-r\Delta t)v_I^0 + o(\Delta t)\Big\}\,.$$

After subtracting v_I^0 from both sides, dividing by $\Delta t > 0$ and taking the limit $\Delta t \searrow 0$ we obtain

$$0 = \max_{e\geq 0}\Big\{V_A^{1,A}(0)e - \tfrac{1}{2}e^2 + V_A^{0,B}(0)e_I^0 - (r+e+e_I^0)v_I^0\Big\}\,.$$

The first order condition becomes $e_I^0 = V_A^{1,A}(0) - v_I^0$, which after substituting yields

$$0 = \tfrac{1}{2}(e_I^0)^2 + e_I^0 V_A^{0,B}(0) - (r+e_I^0)[V_A^{1,A}(0) - e_I^0],$$

which is the equation (13). Its right-hand side is a convex quadratic polynomial of e_I^0, that is negative at $e_I^0 = 0$ and positive at $e_I^0 = V_A^{1,A}(0)$. As a result, the equation has a unique root e_I^0 in the interval $(0, V_A^{1,A}(0))$.

We now show that no firm wants to deviate if and only if inequality (12) holds. Assume that a firm (say, firm A) deviates and does not reveal. Denote by $\tilde{e}^1$ its corresponding optimal effort and by $\tilde{v}_I^1$ its continuation value. By a similar argument as above, after taking the limit $\Delta t \searrow 0$,

$$0 = \max_{e\geq 0}\Big\{1\cdot e - \tfrac{1}{2}e^2 + V_A^{1,B}(0)e_I^0 - (r+e+e_I^0)\tilde{v}_I^1\Big\}\,.$$

The first order condition becomes $\tilde{e}^1 = 1 - \tilde{v}_I^1$, which after substituting yields

$$0 = \tfrac{1}{2}(1-\tilde{v}_I^1)^2 + e_I^0 V_A^{1,B}(0) - (r+e_I^0)\tilde{v}_I^1. \tag{30}$$

The right-hand side of this equation is a quadratic polynomial of $\tilde{v}_I^1$, that is positive at $\tilde{v}_I^1 = 0$ and negative at $\tilde{v}_I^1 = 1$. Thus, there is a unique $\tilde{v}_I^1 \in (0,1)$ that solves the equation (30). The equilibrium condition ensuring that firm A weakly prefers to reveal its breakthrough is $\tilde{v}_I^1 \leq V_A^{1,A}(0)$. This condition holds, if and only if the right-hand side of (30) evaluated at $\tilde{v}_I^1 = V_A^{1,A}(0)$ is non-positive. That gives us the inequality (12). □

D.3 Mixed-strategy Equilibria: Law of Motion

Proof of Lemma 2. Recall that $p_{M,t}^j$ is the posterior probability that firm j is in state 1 at time t from the perspective of firm $-j$. Unlike in the game without the option to reveal, the posterior here is conditioned not only on firm j not having patented but also on its not having revealed by time t, which we denote by the event N_t^j.[29] Accordingly,

$$\begin{aligned} p_{M,t+\Delta t}^j &= P[k_{t+\Delta t}^j = 1 \mid k_{t+\Delta t}^j < 2, N_{t+\Delta t}^j] = \frac{P[k_{t+\Delta t}^j = 1, N_{t+\Delta t}^j \mid k_t^j < 2, N_t^j]}{P[k_{t+\Delta t}^j < 2, N_{t+\Delta t}^j \mid k_t^j < 2, N_t^j]} \\ &= \frac{(1-p_{M,t}^j)e_{M,t}^{0,j}\Delta t + p_{M,t}^j(1-e_{M,t}^{1,j}\Delta t) - g_{M,t}^j \Delta t}{1 - p_{M,t}^j e_{M,t}^{1,j}\Delta t - g_{M,t}^j \Delta t} + o(\Delta t). \end{aligned}$$

Taking derivative with respect to Δt and evaluating at $\Delta t = 0$, we conclude

$$\begin{aligned} \dot{p}_{M,t}^j &= [(1-p_{M,t}^j)e_{M,t}^{0,j} - p_{M,t}^j e_{M,t}^{1,j} - g_{M,t}^j] \cdot 1 + p_{M,t}^j \cdot \left(p_{M,t}^j e_{M,t}^{1,j} + g_{M,t}^j\right) \\ &= (1-p_{M,t}^j)(e_{M,t}^{0,j} - p_{M,t}^j e_{M,t}^{1,j} - g_{M,t}^j). \end{aligned}$$

This concludes the proof. □

Proof of Lemma 3. Consider a time $t \geq 0$ before either of the firms has revealed. First, we show that the hazard rate is finite. Consider a symmetric equilibrium and analyze the situation from the perspective of firm A. If $p_t = 0$, then the claim is trivial as there is nothing to be revealed. In the rest of the proof, consider $p_t > 0$. Distinguish three cases based on how the continuation value $V_A^{1,A}(p_t)$ that firm A (when successful) obtains by revealing compares with the continuation value $V_M^1(p_t)$ that it obtains by not revealing.

Case 1. Let $V_A^{1,A}(p_t) < V_M^1(p_t)$. Then it does not pay off to reveal, and thus $G_M(p_t) = 0$.

[29] In fact, the probability is conditioned on neither firm having patented nor revealed, but this has no impact on the calculation.

Case 2. Let $V_A^{1,A}(p_t) > V_M^1(p_t)$. This case cannot occur in equilibrium as firm A would have strict incentive to reveal already prior to time t.

Case 3. Let $V_A^{1,A}(p_t) = V_M^1(p_t)$. Suppose to the contrary that $g_{M,t} = +\infty$ at time t. Hence, the chance with which firm B reveals in the time interval $[t, t+\Delta t]$ is an arbitrarily large multiple of Δt as Δt goes to 0. We use the argument that waiting then gives firm A an informational advantage. There are two cases of what could happen in the time interval $[t, t+\Delta t]$. If firm B does not reveal in the time interval, then waiting by Δt has only negligible impact on firm A's payoff (of the order $O(\Delta t)$). If firm B reveals in the time interval, then conditioned on firm B having a breakthrough, A's continuation value from revealing at time t is $v^{1,A/B}(p_t)$,[30] while continuation value from waiting is $V_A^{1,B}(p_{t+\Delta t})$. Recall that $v^{1,A/B}(p_t) < V_A^{1,B}(p_t)$ by Proposition 6 due to firm B's informational advantage, and so for Δt small enough $v^{1,A/B}(p_t) < V_A^{1,B}(p_{t+\Delta t})$, meaning that the firm profits from waiting. Thus, firm A prefers to postpone revelation in that case, implying $G_M(p_t) = 0$, which is a contradiction.

Second, we show that p_t is non-decreasing. Since $G_M(p_t)$ is finite, p_t is a continuous function of time (it cannot drop discretely). However, then prior to anyone's revelation p_t can never be decreasing in a Markov perfect Bayesian equilibrium, as otherwise there would be times $t_1 < t_2$ such that $p_{t_1} = p_{t_2}$, but $\dot{p}_{M,t} = (1-p_t)[E_M^0(p_t) - p_t E_M^1(p_t) - G_M(p_t)]$ is positive at t_1 and negative at t_2, impossible. As a result, $\dot{p}_{M,t} \geq 0$ and so $G_M(p_t) \leq E_M^0(p_t) - p_t E_M^1(p_t)$.

Finally, consider an equilibrium. If a firm has the strategy to reveal a breakthrough with certainty at time $t = 0$ (i.e., $g_{M,0} = e_{M,0}^0$), then p_t stays constantly at zero.[31] Indeed, the Markov property then implies that until one of the firms reveals, the firms have to choose the same action at all times because the payoff-relevant state p_t does not change. So if the firm reveals with certainty at time $t = 0$, then the equilibrium has to be the

[30] The value function $v^{1,A/B}$ was introduced in the proof of Proposition 6. Here we use it as a function of the belief p_t.

[31] Recall that $g_{M,t}$ is assumed to be right-continuous. Revealing a breakthrough at time $t = 0$ with certainty means that $g_{M,t}$ converges to $e_{M,0}^0$ as $t \searrow 0$.

instant-revelation equilibrium.

Conversely, if the firm does not have the strategy to reveal with certainty at time $t = 0$, then $\dot{p}_0 > 0$, and since p_t is non-decreasing, it follows that $p_t > 0$ for all $t > 0$, and thus it does not reveal with certainty, as $g_{M,t} \leq e^0_{M,t} - p_t e^1_{M,t} < e^0_{M,t}$. □

D.4 Mixed-strategy Equilibria: Optimal Effort and Value Functions

The continuation value of a successful firm (before anyone has revealed) $v^1_{M,t}$ is given by the following recursive formula:

$$v^1_{M,t} = \max_{e \geq 0} \Big\{ e\Delta t - \tfrac{1}{2}e^2 \Delta t + V^{1,B}_A(p_{M,t}) g_{M,t} \Delta t \\ + \left[1 - (r + e + g_{M,t} + p_{M,t} e^1_{M,t})\Delta t\right] v^1_{M,t+\Delta t} + o(\Delta t) \Big\}.$$

This formula differs from those in previous sections by including the term $V^{1,B}_A(p_{M,t}) g_{M,t} \Delta t$, which captures the event where the rival reveals during $[t, t + \Delta t]$ (occurring with probability $g_{M,t}$) and yields continuation value $V^{1,B}_A(p_{M,t})$. The formula assumes the firm itself does not reveal at time t. This assumption is without loss of generality: in equilibrium, when the firm is mixing (randomizing over revelation), it must be indifferent between revealing and not revealing, so whether we evaluate the value under the assumption of revealing or not revealing makes no difference to the calculated value.

After subtracting $v^1_{M,t+\Delta t}$, dividing by Δt, and taking the limit $\Delta t \searrow 0$, we obtain

$$-\dot{v}^1_{M,t} = \max_{e \geq 0} \left\{ e - \tfrac{1}{2}e^2 + V^{1,B}_A(p_{M,t}) g_{M,t} - (r + e + g_{M,t} + p_{M,t} e^1_{M,t}) v^1_{M,t} \right\}.$$

The first order condition yields $e^1_{M,t} = 1 - v^1_{M,t}$. Plugging it back yields the equation (14). The derivation of the equation (15) for $\dot{v}^0_{M,t}$ is analogous, and the equation (16) for $\dot{p}_{M,t}$ follows from Lemma 2.

The following lemma establishes the existence and uniqueness of the crossing point $\overline{p}_M$

that determines the mixed-revelation equilibrium. When the no-revelation equilibrium does not exist (i.e., $\rho > \rho_N$), we have $V_A^{1,A}(0) > V_S^1(0)$. At the same time, Lemma D.1 guarantees that $V_A^{1,A}(p_{S,*}) < V_S^1(p_{S,*})$ (provided $\rho < \rho_I$). By continuity, the two value functions must cross, and the lemma shows that this crossing is unique.

Lemma D.3 (partially numerical). *If $V_A^{1,A}(0) > V_S^1(0)$ and $\rho < \rho_I$, then there exists a unique $\overline{p}_M \in (0, p_{S,*})$ such that $V_A^{1,A}(\bar{p}) = V_S^1(\bar{p})$ and $V_A^{1,A}(p) > V_S^1(p)$ if and only if $p < \bar{p}$.*

The proof of Proposition 11 is based on the following lemma.

Lemma D.4 (partially numerical). *In any symmetric equilibrium other than the instant-revelation equilibrium the following statements hold:*

(i) If $\lim_{s \nearrow t} g_{M,s} = 0$, then $g_{M,t} = 0$.

(ii) There exists some $T_0 > 0$ such that $g_{M,t} = 0$ for all $t \geq T_0$.

(iii) Let $T = \inf\{T_0 : g_{M,t} = 0$ for all $t \geq T_0\}$. Then $V_A^{1,A}(p_{M,T}) = V_S^1(p_{M,T})$.

(iv) $g_{M,t}$ is strictly decreasing, or it is equal to 0.

Proof of Proposition 11. Consider T as defined in Lemma D.4 (iii). Since the equilibrium involves randomization, $T > 0$.

(i) As stated in the proof of Lemma D.4 (iii), $V_A^{1,A}(p_{M,T}) = V_S^1(p_{M,T})$ follows from the continuity of firm's continuation value in a specific state. Necessarily $p_{M,T} < p_{S,*}$, as otherwise $p_{S,t}$ would be decreasing for $t > T$, which is not possible (due to Lemma 3).

(ii) The necessary condition for a firm to be randomizing over revelation in equilibrium, i.e., for $0 < g_{M,t} < e^0_{M,t}$, is that $R_t(g_{M,t}) = 0$. Solving for $g_{M,t}$ yields the condition (19). The statement that $g_{M,t}$ is decreasing follows from Lemma D.4 (iv).

(iii) The statement follows directly from Lemma D.4 (ii)–(iii). □

D.5 Equilibrium Characterization

Proof of Proposition 12 (partially numerical). By Proposition 11 the mixed-revelation equilibrium is the unique candidate for equilibrium involving mixed strategies. Next we show that mixed-revelation equilibrium exists if and only if neither of the pure-strategy equilibria exists. We show numerically that $V_A^{1,A}(p_{S,*}) < V_S^1(p_{S,*})$.

First, no-revelation equilibrium exists if and only if $V_A^{1,A}(p) \leq V_S^1(p)$ for all $p \in [0, p_{S,*})$ (Proposition 9), and so there exists $p_{M,T} = \overline{p}_M \in [0, p_{S,*})$ satisfying condition (i) of Proposition 11 if and only if no-revelation equilibrium does not exist.

Second, assuming there does not exist no-revelation equilibrium, we solve for $G_M(p)$ and $E_M^0(p)$ on $p \in [0, \bar{p}]$ using the characterization of mixed-revelation equilibria from Proposition 11. We find that whenever instant-revelation equilibrium exists, then $G_M(0) \geq E_M^0(0)$ and so no mixed-revelation equilibrium exists. Otherwise, the condition $G_M(p) \leq E_M^0(p) - pE_A^{1,A}(p)$ is satisfied for all $p \in [0, \bar{p}]$ and a unique mixed-revelation equilibrium is found.

As a result, there always exists a unique equilibrium. Its type depends on the parameters. The thresholds ρ_N and ρ_I are found numerically. □

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