Supplementary Appendix to

Revealing Private Information in a Patent Race

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A Supplementary Appendix: Proofs for Appendix A

Throughout this section we drop the subscript C.

Proof of Lemma A.1. First, $v^{k+1,l} > v^{k,l}$ holds trivially as $e^{k,l} > 0$ by assumption. The inequality $v^{k,l+1} < v^{k,l}$ holds trivially for $l = 1$ (since $v^{k,2} = 0$); it remains to prove it for $l = 0$. Use mathematical induction: show that the weak inequality $v^{k+1,1} \leq v^{k+1,0}$ implies the strict inequality $v^{k,1} < v^{k,0}$. We have $v^{21} \leq v^{20}$ as both values equal 1. This initializes the induction for $k = 1$, establishing the base case. Consider $k \in \{0, 1\}$ and assume that $v^{k+1,1} \leq v^{k+1,0}$.

Then we have

$$
\begin{aligned}
0 &= \max_{e\in[0,1]} \left\{ e \cdot v^{k+1,0} - \tfrac{1}{2}e^2 + e^{0,k} \cdot (v^{k,1} - v^{k,0}) - (r + e) \cdot v^{k,0} \right\} \\
&\geq e^{k,1} \cdot v^{k+1,0} - \tfrac{1}{2}(e^{k,1})^2 + e^{0,k} \cdot (v^{k,1} - v^{k,0}) - (r + e^{k,1}) \cdot v^{k,0} \\
&\geq e^{k,1} \cdot v^{k+1,1} - \tfrac{1}{2}(e^{k,1})^2 + e^{0,k} \cdot (v^{k,1} - v^{k,0}) - (r + e^{k,1}) \cdot v^{k,0} \\
&= e^{k,1} \cdot v^{k+1,1} - \tfrac{1}{2}(e^{k,1})^2 - (r + e^{k,1}) \cdot v^{k,1} + (r + e^{k,1} + e^{0,k}) \cdot (v^{k,1} - v^{k,0}) \\
&> \underbrace{e^{k,1} \cdot v^{k+1,1} - \tfrac{1}{2}(e^{k,1})^2 + e^{1,k} \cdot (v^{k,2} - v^{k,1}) - (r + e^{k,1}) \cdot v^{k,1}}_{=0} + (r + e^{k,1} + e^{0,k}) \cdot (v^{k,1} - v^{k,0}) \\
&= (r + e^{k,1} + e^{0,k}) \cdot (v^{k,1} - v^{k,0}),
\end{aligned}
$$

and so $0 > v^{k,1} - v^{k,0}$. We conclude that $v^{k,1} < v^{k,0}$ for $k \in \{0, 1\}$. □

Proof of Lemma A.2. We prove it recursively by decreasing k and l. For $k = 2$ or $l = 2$ the uniqueness is trivial. Take any $k, l \in \{0, 1\}$ for which the uniqueness of $v^{k+1,l}, v^{k,l+1}, v^{l+1,k}, v^{l,k+1}$ has been proven already (initially it is the case for $k = l = 1$). Separating $e^{l,k} = v^{l+1,k} - v^{l,k}$ in the equation (22),

$$v^{l,k} = v^{l+1,k} - e^{l,k} = v^{l+1,k} - \frac{\frac{1}{2}(v^{k+1,l} - v^{k,l})^2 - rv^{k,l}}{v^{k,l} - v^{k,l+1}} = \gamma^{l,k}(v^{k,l}),$$
$$\text{where} \quad \gamma^{l,k}(z) := v^{l+1,k} - \frac{\frac{1}{2}(v^{k+1,l} - z)^2 - rz}{z - v^{k,l+1}}$$

so that $v^{l,k}$ is expressed as a function of $v^{k,l}$ and other variables which are already known to be uniquely defined. Notice that since $v^{k,l+1} \leq v^{k+1,l+1} < v^{k+1,l}$ (Lemma A.1), we have

$$\tfrac{1}{2}(v^{k+1,l} - v^{k,l+1})^2 - rv^{k,l+1} > \tfrac{1}{2}(v^{k+1,l+1} - v^{k,l+1})^2 - rv^{k,l+1} \geq 0,$$

where the last inequality $\frac{1}{2}(v^{k+1,l+1} - v^{k,l+1})^2 - rv^{k,l+1} \geq 0$ follows from equation (22) applied to state $(k, l+1)$ with $e^{l+1,k} > 0$; and so the strictly decreasing function $z \mapsto \frac{1}{2}(v^{k+1,l} - z)^2 - rz$ has unique root on the interval $(v^{k,l+1}, v^{k+1,l})$; denote it $\overline{v}^{k,l}$.

It follows that $\gamma^{l,k}(z)$ is a strictly increasing function on the interval $(v^{k,l+1}, \overline{v}^{k,l}]$. Moreover,

$$\begin{aligned}\gamma^{l,k}(z) &= v^{l+1,k} - \frac{\frac{1}{2}z^2 - v^{k+1,l}z + \frac{1}{2}(v^{k+1,l})^2 - rz}{z - v^{k,l+1}} \\ &= v^{l+1,k} - \tfrac{1}{2}v^{k,l+1} + v^{k+1,l} + r - \tfrac{1}{2}z - \frac{\frac{1}{2}(v^{k+1,l} - v^{k,l+1})^2 - rv^{k,l+1}}{z - v^{k,l+1}}.\end{aligned}$$

Since the term in the numerator is positive, function $\gamma^{l,k}(z)$ is concave. In summary, $\gamma^{l,k}(z)$ is a continuous, concave, strictly increasing function on the interval $(v^{k,l+1}, \overline{v}^{k,l}]$ with range from $-\infty$ to $v^{l+1,k}$.

By symmetry, there is a continuous, concave, strictly increasing function $\gamma^{k,l}(z)$ defined on the interval $(v^{l,k+1}, \overline{v}^{l,k}]$ with range from $-\infty$ to $v^{k+1,l}$, such that $v^{k,l} = \gamma^{k,l}(v^{l,k})$. As illustrated in Figure 1 it should be clear that there is a unique point $(v^{k,l}, v^{l,k}) \in (v^{k,l+1}, \overline{v}^{k,l}] \times (v^{l,k+1}, \overline{v}^{l,k}]$ that satisfies both $v^{l,k} = \gamma^{l,k}(v^{k,l})$ and $v^{k,l} = \gamma^{k,l}(v^{l,k})$. □

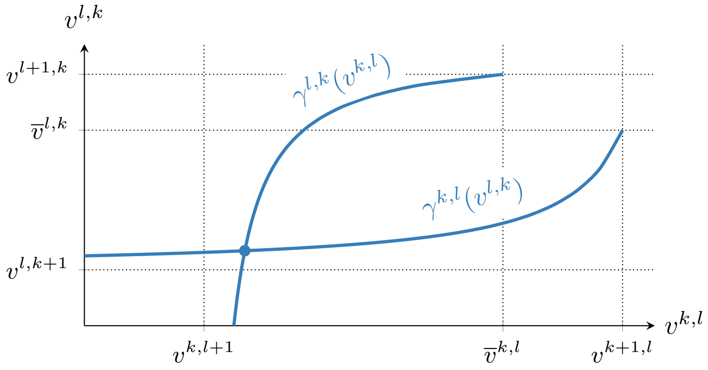


Figure 1: Illustration of the uniqueness of $v^{k,l}$ and $v^{l,k}$ as an intersection of the graphs of reaction functions.

Proof of Lemma A.3. Let

$$\begin{aligned}
\zeta^{11}(z) &:= \tfrac{3}{2}z^2 + (r-1)z - r,\\
\zeta^{10}(z) &:= \tfrac{1}{2}z^2 + (r+e^{01})z - e^{01}e^{11} - r,\\
\zeta^{01}(z) &:= \tfrac{1}{2}z^2 + (r+e^{10})z - (r+e^{10})(1-e^{11}),\\
\zeta^{00}(z) &:= \tfrac{3}{2}z^2 + (r-e^{11}+e^{10}-e^{01})z - r(1-e^{10}).
\end{aligned}$$

Then $e^{k,l}$ satisfies $\zeta^{k,l}(e^{k,l}) = 0$ for all $k,l \in \{0,1\}$. In addition, $\zeta^{k,l}(z) < 0$ when $0 \le z < e^{k,l}$, and $\zeta^{k,l}(z) > 0$ when $z > e^{k,l}$, for all $k,l \in \{0,1\}$. Indeed, using the optimality condition $e^{k,l} = v^{k+1,l} - v^{k,l}$, we can express the values in terms of efforts: $v^{11} = 1 - e^{11}$, $v^{10} = 1-e^{10}$, $v^{01} = v^{11} - e^{01} = 1 - e^{11} - e^{01}$, and $v^{00} = v^{10} - e^{00} = 1 - e^{10} - e^{00}$. Substituting those values into (22), we obtain that equation (22) can be written as $\zeta^{k,l}(e^{k,l}) = 0$, for all $k,l \in \{0,1\}$. Observe that for any $k,l \in \{0,1\}$, the quadratic polynomial $\zeta^{k,l}(z)$ has a positive leading coefficient and a negative intercept, and so it has a unique positive root. Hence, $e^{k,l}$ is the root, and $\zeta^{k,l}(z)$ is negative to the left of the root, and positive to the right of it.

(i) Evaluating ζ^{11} at the lower and upper estimate of e^{11},

$$\zeta^{11}(\underline{E}^{11}) = -\frac{r}{(2r+3)^2} < 0, \quad \text{and} \quad \zeta^{11}(\overline{E}^{11}) = \frac{3}{8(r+2)^2} > 0.$$

The former yields $\underline{E}^{11} < e^{11}$, whereas the latter yields $e^{11} < \overline{E}^{11}$.

(ii) Applying the result of (i), $e^{01} = v^{11} - v^{01} < v^{11} = 1 - e^{11} < 1 - \underline{E}^{11} = \frac{1}{3+2r}$.

(iii) We have

$$
\begin{aligned}
\zeta^{10}(\overline{E}^{10}) &= \tfrac{1}{2}(\overline{E}^{10})^2 + r\overline{E}^{10} + e^{01}(\overline{E}^{10} - e^{11}) - r \\
&> \tfrac{1}{2}(\overline{E}^{10})^2 + r\overline{E}^{10} + e^{01}(\overline{E}^{10} - \overline{E}^{11}) - r \\
&> \tfrac{1}{2}(\overline{E}^{10})^2 + r\overline{E}^{10} + \overline{E}^{01}(\overline{E}^{10} - \overline{E}^{11}) - r \\
&= \frac{2 + 3r + 2r^2}{2(2+2r)^2(2r^2+7r+6)} > 0,
\end{aligned}
$$

where the first inequality follows from (i) and the second inequality from (ii) and $\overline{E}^{10} - \overline{E}^{11} = -1/[(2+2r)(2+r)] < 0$. Thus, indeed $\overline{E}^{10} > e^{10}$.

(iv) For $r \geq 1$, we have

$$
\begin{aligned}
\zeta^{00}(\underline{E}^{00}) &= \tfrac{3}{2}(\underline{E}^{00})^2 + \underline{E}^{00}(r - e^{11} + e^{10} - e^{01}) - r(1 - e^{10}) \\
&< \tfrac{3}{2}(\underline{E}^{00})^2 + \underline{E}^{00}(r - \underline{E}^{11} + \overline{E}^{10}) - r(1 - \overline{E}^{10}) \\
&= \frac{1-r}{(3+2r)^2} \leq 0,
\end{aligned}
$$

where the first inequality follows from (i), (iii), and $e^{01} > 0$.

For $r \in (0,1)$, let $f(r) := \zeta^{00}(\underline{E}^{00})$. At $r = 0$, we have $e^{11} = \frac{2}{3}$ (from ζ^{11}), and the coupled system for e^{10}, e^{01} reduces to a single equation for the ratio $q := e^{10}/e^{01}$:

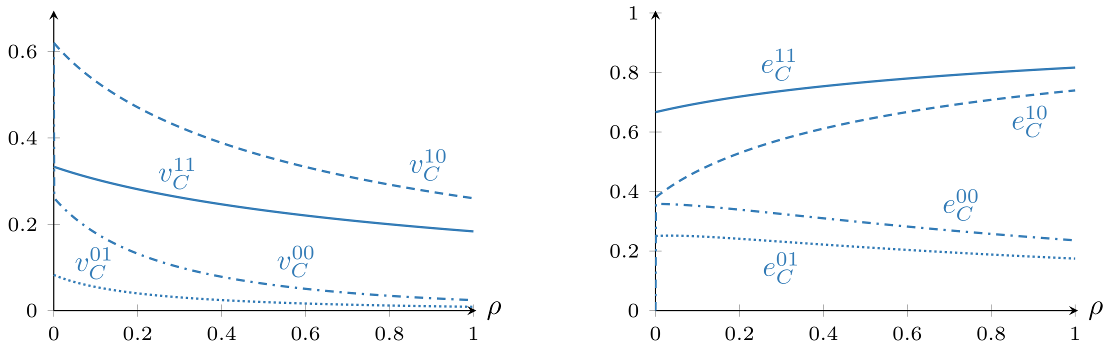
$$
q^3 + 2q^2 - 4q - 2 = 0.
$$

Substituting, one obtains $f(0) = (4q_0^2 - 6q_0 - 1)/[18(1+2q_0)]$, where q_0 is the unique positive root of the cubic. Since the cubic is increasing for $q > \frac{2}{3}$ and evaluates to a positive value at $q^* := (3+\sqrt{13})/4$ (the positive root of $4q^2 - 6q - 1$), we get $q_0 < q^*$ and hence $f(0) < 0$. Moreover, f is decreasing on $(0,1)$, as can be verified by implicit differentiation of the equilibrium system. Therefore $f(r) \leq f(0) < 0$ for all $r \in (0,1]$.

This implies $\underline{E}^{00} < e^{00}$. □

B Supplementary Appendix: Proofs for Section 4 (Patent Race with Unobservable Progress)

Proof of Lemma 1. The posterior belief follows the Bayes Law. Take the conditioned probability p_t^j as given and assume that the game has not ended by time t. Then with probability $(1-p_t^j)$ the state is $k_t^j = 0$, and with hazard rate $e_t^{0,j}$ it proceeds to the state $k_{t+\Delta t}^j = 1$; with probability p_t^j the state is $k_t^j = 1$ and with the hazard rate $e_t^{1,j}$ it proceeds



(a) (b)

Figure 2: Continuation values and efforts in the four different states of the complete information version of the game as a function of the research difficulty ρ.

and the game ends. Then

$$p^j_{t+\Delta t} = \Pr[k^j_{t+\Delta t} = 1 \mid k^j_{t+\Delta t} < 2] = \frac{\Pr\left[k^j_{t+\Delta t} = 1 \mid k^j_t < 2\right]}{\Pr\left[k^j_{t+\Delta t} < 2 \mid k^j_t < 2\right]}$$
$$= \frac{(1-p^j_t)e^{0,j}_t \Delta t + p^j_t(1 - e^{1,j}_t \Delta t) + o(\Delta t)}{1 - p^j_t e^{1,j}_t \Delta t + o(\Delta t)},$$

and so

$$\dot{p}^j_t = \frac{\partial}{\partial \Delta t} p^j_{t+\Delta t}\bigg|_{\Delta t = 0} = (1-p^j_t)(e^{0,j}_t - p^j_t e^{1,j}_t).$$

This completes the proof. □

C Supplementary Appendix: Proofs for Appendix B

Proof of Lemma B.1. Multiplying each equation in the system of ODEs (24)–(26) by a/v and dividing the first two by v yields:

$$-a\frac{\dot{v}^{1,j}_t}{v^2} = \frac{1}{2}\left(\frac{ae^{1,j}_t}{v}\right)^2 - \left(\frac{ar}{v} + p^{-j}_t \frac{ae^{1,-j}_t}{v}\right)\frac{v^{1,j}_t}{v}$$
$$-a\frac{\dot{v}^{0,j}_t}{v^2} = \frac{1}{2}\left(\frac{ae^{0,j}_t}{v}\right)^2 - \left(\frac{ar}{v} + p^{-j}_t \frac{ae^{1,-j}_t}{v}\right)\frac{v^{0,j}_t}{v}$$
$$\frac{a\dot{p}^j_t}{v} = (1-p^j_t)\left(\frac{ae^{0,j}_t}{v} - p^j_t\frac{ae^{1,j}_t}{v}\right).$$

This system is equivalent to (24)–(26) with parameters $\hat{v} = 1$, $\hat{a} = 1$, and $\rho = ar/v$, and transformed variables

$$\hat{v}_t^{k,j} = \frac{v_{ta/v}^{k,j}}{v}, \quad \hat{e}_t^{k,j} = \frac{ae_{ta/v}^{k,j}}{v}, \quad \text{and} \quad \hat{p}_t^j = p_{ta/v}^j,$$

where $k \in \{0, 1\}$, $j \in \{A, B\}$. □

D Supplementary Appendix: Proofs for Appendix C

Throughout this section we study symmetric equilibria, therefore we omit $j \in \{A, B\}$ from the superscript and use the normalization $v = 1$ and $a = 1$.

We state some elementary properties of a function that will be used in numerous proofs.

Lemma D.1. *For any given $r > 0$, the function $\psi : [0, 1) \to [-r, +\infty)$, defined by*

$$\psi(z) := \frac{z^2}{2(1 - z)} - r \tag{1}$$

is strictly increasing and strictly convex, and it admits a well-defined inverse function $\psi^{-1} : [-r, +\infty) \to [0, 1)$ that is strictly increasing and strictly concave. In addition, ψ has a unique positive fixed point z_*. Moreover, $z_* \in (\frac{2}{3}, 1)$, $\psi(z) < z$ for all $z \in [0, z_*)$, and $\psi(z) > z$ for all $z \in (z_*, 1)$.

Proof. We first establish the properties of ψ. Both $z \mapsto z^2$ and $z \mapsto 1/(1 - z)$ are strictly positive (except at $z = 0$), strictly increasing, and strictly convex on $[0, 1)$. Their product, $z^2/(1 - z)$, thus shares these properties, and so does $\psi(z) = \frac{z^2}{2(1-z)} - r$.

We have $\psi(0) = -r < 0$ and $\lim_{z \nearrow 1} \psi(z) = +\infty$. By continuity and strict monotonicity, ψ is invertible, and its inverse ψ^{-1} is well defined and strictly increasing. Since ψ is strictly convex, ψ^{-1} is strictly concave.

From the above, ψ has a unique fixed point z_*, as it crosses the 45-degree line at exactly one point. This implies $\psi(z) < z$ if and only if $z < z_*$. Finally, evaluating at $z = \frac{2}{3}$ gives $\psi(\frac{2}{3}) = \frac{2}{3} - r < \frac{2}{3}$, which establishes $z_* > \frac{2}{3}$. □

Proof of Lemma C.1. A critical point is characterized by the condition $(\dot{e}_{S,t}^1, \dot{e}_{S,t}^0, \dot{p}_{S,t}) = (0, 0, 0)$. Dropping the subscripts, we obtain $e^0 = pe^1$ and

$$0 = \tfrac{1}{2}(e^1)^2 - (r + e^0)(1 - e^1), \tag{2}$$

$$0 = \tfrac{1}{2}(e^0)^2 - \tfrac{1}{2}(e^1)^2 + (r + e^0)e^0. \tag{3}$$

Under the assumption that $e^1 \in (0,1)$ and $p \in (0,1)$, the above system is equivalent to[1]

$$e^0 = \frac{(e^1)^2}{2(1-e^1)} - r, \tag{4}$$

$$1 = \left(\frac{e^0}{e^1}\right)^2 + \frac{e^0}{1-e^1}. \tag{5}$$

Define the function Φ on the interval $(0,1)$ by

$$\Phi(z) := \left[\frac{\psi(z)}{z}\right]^2 + \frac{\psi(z)}{1-z},$$

where the function ψ is defined by (1). Then the critical point is characterized by the system of equations $e^0 = \psi(e^1)$, $1 = \Phi(e^1)$, and $p = e^0/e^1$. Recall that by Lemma D.1 the function ψ is strictly increasing. The function Φ is strictly increasing on the interval $[\psi^{-1}(0), 1)$. To see this, note that both $\psi(z)/(1-z)$ and $\psi(z)/z$ are strictly increasing functions of z on this interval, since $\psi(z) \geq 0$ there. Moreover, $\Phi\left(\psi^{-1}(0)\right) = 0$, and

$$\Phi(z_*) > [\psi(z_*)/z_*]^2 = 1,$$

where z_* is the fixed point of ψ. It follows that the inverse function Φ^{-1} is well defined on $[0,1]$.

We conclude by assigning $e^1 = \Phi^{-1}(1) \in (\psi^{-1}(0), 1) \subset (0,1)$, $e^0 = \psi(e^1)$, and $p = e^0/e^1$. Since $1 < \Phi(z_*)$, $e^1 < z_*$, and so by Lemma D.1, $e^0 = \psi(e^1) < e^1$; moreover, $e^0 > 0$ since $e^1 > \psi^{-1}(0)$. Consequently, $p = e^0/e^1 \in (0,1)$. The tuple (e^1, e^0, p) is the unique critical point of the system of ODEs (27)–(29) with the desired properties. □

Before proceeding with the proof of Lemma C.2, we establish some useful inequalities among the variables at the critical point (Lemma D.2).[2]

Lemma D.2. *At the unique critical point $(e^1, e^0, p) = (e^1_{S,*}, e^0_{S,*}, p_{S,*})$ the following statements hold:*

(i) $e^1 > \frac{1}{2} > v^1$;

(ii) $e^0 > v^1 e^1$;

(iii) $r + e^0 > \frac{1}{2} e^1$.

[1] The equation (5) is obtained by substituting for $(r + e^0)$ in (3) from (2), and dividing the equation by $\frac{1}{2}(e^1)^2 > 0$.

[2] In this section we drop the subscript for the values of the variables at the critical point; specifically $v^1 = 1 - e^1$ represents $v^1_{S,*}$.

Proof. (i) If $r \geq \frac{1}{4}$, then $\psi(\frac{1}{2}) = \frac{1}{4} - r \leq 0$, and so $\psi^{-1}(0) \geq \frac{1}{2}$. Since $e^1 > \psi^{-1}(0)$ by Lemma C.1, we get $e^1 > \frac{1}{2}$. If $r < \frac{1}{4}$, then $\psi(\frac{1}{2}) = \frac{1}{4} - r \in (0, \frac{1}{4})$, and so

$$\Phi\left(\tfrac{1}{2}\right) = \left[\frac{\psi(\frac{1}{2})}{\frac{1}{2}}\right]^2 + \frac{\psi(\frac{1}{2})}{\frac{1}{2}} < \left(\frac{\frac{1}{4}}{\frac{1}{2}}\right)^2 + \frac{\frac{1}{4}}{\frac{1}{2}} = \tfrac{1}{4} + \tfrac{1}{2} < 1,$$

and so $e^1 = \Phi^{-1}(1) > \frac{1}{2}$. In either case, $v^1 = 1 - e^1 < \frac{1}{2}$.

(ii) If $e^0 \leq e^1 v^1 = e^1(1 - e^1)$, then the equation (5) would lead to a contradiction

$$1 = \left(\frac{e^0}{e^1}\right)^2 + \frac{e^0}{1 - e^1} \leq (1 - e^1)^2 + e^1 < (1 - e^1) + e^1 = 1.$$

(iii) Applying the result of (i),

$$r + e^0 = \frac{(e^1)^2}{2(1 - e^1)} = \frac{e^1}{1 - e^1} \cdot \frac{e^1}{2} > \frac{e^1}{2}.$$

This completes the proof. □

Proof of Lemma C.2. Consider the critical point $(e^1, e^0, p) = (e^1_{S,*}, e^0_{S,*}, p_{S,*})$ and define $w := r + 2e^0$ and $h := e^1 - e^0$. The Jacobian of the system (27)–(29) is equal to (recall that $e^0 = pe^1$ at the critical point)[3]

$$J = \begin{bmatrix} w + e^1 - p & 0 & -v^1 e^1 \\ -e^1 + pe^0 & w & e^0 e^1 \\ -p(1-p) & 1 - p & -h \end{bmatrix}.$$

Eigenvalues λ of J are the roots of the polynomial $Q(\lambda) := |J - \lambda I|$, where I is the identity matrix. We obtain

$$Q(\lambda) = (w + e^1 - p - \lambda)\big[(w - \lambda)(-h - \lambda) - e^0 h\big] + v^1 h\big[e^1 - p(w + e^0 - \lambda)\big]. \tag{6}$$

We can express the polynomial in terms of its coefficients as $Q(\lambda) = -\lambda^3 + b_2\lambda^2 - b_1\lambda + b_0$. Then, by Lemma D.2 (i), $p = e^0/e^1 < 2e^0$, and so

$$b_2 = 2w + e^1 - p - h = 2r + 5e^0 - p > 2r + 3e^0 > 0.$$

[3] The Jacobian entries are computed from $\partial F^k_S/\partial e^l$ and $\partial F^k_S/\partial p$ evaluated at the critical point, using equations (27)–(29).

Next, using inequalities from Lemma D.2 (i) and (ii),

$$\begin{aligned} b_0/h = Q(0)/h &= -(w+e^1-p)(w+e^0)+e^1v^1-p(w+e^0)v^1 \\ &= -\big[w+e^1-p(1-v^1)\big](w+e^0)+e^1v^1 \\ &= -(r+e^0+e^1)(r+3e^0)+e^1v^1 \\ &< -\tfrac{1}{2}(r+3e^0)+e^0<0. \end{aligned}$$

Since $Q(0)<0$ and $Q(\lambda)\to+\infty$ as $\lambda\to-\infty$, the polynomial $Q(\lambda)$ has at least one negative root; denote it λ_1. It remains to prove that the other two (complex) roots λ_2,λ_3 have positive real parts. According to *Vieta's formulas*, $b_2=\lambda_1+\lambda_2+\lambda_3$ and $b_0=\lambda_1\lambda_2\lambda_3$. Then $\lambda_2+\lambda_3=b_2-\lambda_1>0$ and $\lambda_2\lambda_3=b_0/\lambda_1>0$. If the roots λ_2 and λ_3 are real numbers, then they are necessarily positive. If the roots have nonzero imaginary parts, then they must be complex conjugates of each other, and thus have positive real parts. □

Lemma D.3. *Let $\omega^t:[0,1]\to\mathbb{R}$, $t\in\mathbb{R}_+$ be a system of continuously differentiable functions that uniformly converge to some continuous function $\omega^\infty:[0,1]\to\mathbb{R}$ as $t\to+\infty$. Assume that ω^∞ has a unique root z_∞ and that $(\omega^\infty)'(z_\infty)>0$. If $\dot z_t=\omega^t(z_t)$, where $z_t\in[0,1]$, for all $t\ge0$, then $z_t\to z_\infty$ as $t\to+\infty$.*

Proof. Consider a fixed $\varepsilon>0$. Define $L_\varepsilon=\frac{1}{2}\cdot\min\{|\omega^\infty(z)| : z\in[0,1],\ |z-z_\infty|\ge\varepsilon\}$. Since ω^∞ is continuous with a unique root z_∞, L_ε is well defined and positive.

Since the functions ω^t converge uniformly to ω^∞, there exists $\tau_\varepsilon\ge0$ such that $|\omega^t(z)-\omega^\infty(z)|<L_\varepsilon$ for all $z\in[0,1]$ and $t\ge\tau_\varepsilon$.

Using the triangle inequality, we conclude that if $|z_t-z_\infty|\ge\varepsilon$ and $t\ge\tau_\varepsilon$, then

$$\big|\omega^t(z_t)\big|\ge|\omega^\infty(z_t)|-\big|\omega^\infty(z_t)-\omega^t(z_t)\big|\ge 2L_\varepsilon-L_\varepsilon=L_\varepsilon.$$

Specifically, given that $(\omega^\infty)'(z_\infty)>0$, $\omega^t(z_t)<-L_\varepsilon$ for $z_t<z_\infty-\varepsilon$ and $\omega^t(z_t)>L_\varepsilon$ for $z_t>z_\infty+\varepsilon$. Since $\dot z_t=\omega^t(z_t)$, it follows that if $z_{t_0}\le z_\infty-\varepsilon$ for some $t_0\ge\tau_\varepsilon$, then $z_t\le z_\infty-\varepsilon-L_\varepsilon\cdot(t-t_0)$ for all $t\ge t_0$, and eventually z_t goes out of bounds. Similarly, if $z_{t_0}\ge z_\infty+\varepsilon$ for some $t_0\ge\tau_\varepsilon$, then $z_t\ge z_\infty+\varepsilon+L_\varepsilon\cdot(t-t_0)$ for all $t\ge t_0$, and eventually z_t goes out of bounds. We conclude that for every $\varepsilon>0$ and every $t\ge\tau_\varepsilon$, $|z_t-z_\infty|<\varepsilon$, and so indeed $z_t\to z_\infty$ as $t\to+\infty$. □

Proof of Lemma C.3. The Markov property requires that all equilibrium strategies and dynamics depend only on the current belief $p_{S,t}$. Thus, if $p_{S,t_1}=p_{S,t_2}$ at distinct times, we must have $\dot p_{S,t_1}=\dot p_{S,t_2}$. Therefore, $p_{S,t}$ cannot double back or form plateaus except at the critical point, so $p_{S,t}$ is necessarily monotone. Since $p_{S,t}$ is monotone on a bounded range, it converges to some value $p_{S,\infty}$.

The ODEs (27)–(28) for $e^1_{S,t}$ and $e^0_{S,t}$ can be written as

$$\dot{e}^1_{S,t} = \kappa^{1,t}_S(e^1_{S,t}), \qquad \text{where} \qquad \kappa^{1,t}_S(z) := \tfrac{1}{2}z^2 - (r + p_{S,t}z)(1-z),$$
$$\dot{e}^0_{S,t} = \kappa^{0,t}_S(e^0_{S,t}), \qquad \text{where} \qquad \kappa^{0,t}_S(z) := \tfrac{1}{2}z^2 - \tfrac{1}{2}(e^1_{S,t})^2 + (r + p_{S,t}e^1_{S,t})z.$$

The functions $\kappa^{1,t}_S(z)$ are continuously differentiable and converge uniformly to $\kappa^{1,\infty}_S(z) = \frac{1}{2}z^2 - (r + p_{S,\infty}z)(1-z)$ as $t \to +\infty$. Since $\kappa^{1,\infty}_S$ is a quadratic function with a positive leading coefficient, negative intercept, and $\kappa^{1,\infty}_S(1) > 0$, it has a unique positive root $e^1_{S,\infty} \in (0,1)$ and $(\kappa^{1,\infty}_S)'(e^1_{S,\infty}) > 0$. Applying Lemma D.3, we conclude that $e^1_{S,t} \to e^1_{S,\infty}$.

Similarly, the functions $\kappa^{0,t}_S(z)$ are continuously differentiable and converge uniformly to $\kappa^{0,\infty}_S(z) = \frac{1}{2}z^2 - \frac{1}{2}(e^1_{S,\infty})^2 + (r + p_{S,\infty}e^1_{S,\infty})z$ as $t \to +\infty$. Since $\kappa^{0,\infty}_S$ is a quadratic polynomial with a positive leading coefficient, negative intercept, and $\kappa^{0,\infty}_S(e^1_{S,\infty}) > 0$, it has a unique positive root $e^0_{S,\infty} \in (0, e^1_{S,\infty})$, and $(\kappa^{0,\infty}_S)'(e^0_{S,\infty}) > 0$. Applying Lemma D.3, we conclude that $e^0_{S,t} \to e^0_{S,\infty}$.

Since $e^0_{S,\infty} < e^1_{S,\infty}$, it follows from (29) that $p_{S,\infty} < 1$, because otherwise $\dot{p}_{S,t}$ would necessarily be negative for t large, which would prevent it from reaching $p_{S,\infty}$. We conclude that $(e^1_{S,\infty}, e^0_{S,\infty}, p_{S,\infty})$ is a critical point of the system (27)–(29) with $p_{S,\infty} < 1$, and thus, by Lemma C.1, $(e^1_{S,\infty}, e^0_{S,\infty}, p_{S,\infty}) = (e^1_{S,*}, e^0_{S,*}, p_{S,*})$.

Finally, provided that $p_{S,0} < p_{S,*}$, $p_{S,t}$ is necessarily increasing. □

Proof of Lemma C.5. As the first step, we show that $\lambda_1 < -h = -(e^1 - e^0)$. It follows from (6) and $w = r + 2e^0$ that

$$Q(-h) = -(r + e^0 + 2e^1 - p)e^0 h + v^1 h \cdot \left[e^1 - p(r + 2e^0 + e^1)\right].$$

By Lemma D.2 (i), $p = e^0/e^1 < 2e^0 < 2e^1$ and $e^0 = pe^1 > pv^1$. Thus, $(r+e^0+2e^1-p)e^0 > re^0 > rpv^1$ and we obtain

$$\begin{aligned} Q(-h) \cdot \frac{e^1}{hv^1} &< -rpe^1 + e^1\left[e^1 - p(r + 2e^0 + e^1)\right] \\ &= (e^1)^2 - p(e^1)^2 - 2pe^1(r + e^0) \\ &= (e^1)^2 - e^0e^1 - 2e^0(r + e^0) \\ &< (e^1)^2 - (e^0)^2 - 2(r + e^0)e^0 = 0, \end{aligned}$$

where the last equality follows from (3). Since the polynomial $Q(\lambda)$ has a unique negative root, $Q(-h) < 0$, and $Q(\lambda) \to +\infty$ as $\lambda \to -\infty$, thus indeed $\lambda_1 < -h = -(e^1 - e^0)$.

The eigenvector associated with λ_1 is characterized by the vector equation $(J - \lambda_1 I)\mu_S = 0$, which is equivalent to

$$\begin{aligned}(w + e^1 - p - \lambda_1)\mu_S^1 - v^1 e^1 \mu_S^p &= 0,\\ -(e^1 - pe^0)\mu_S^1 + (w - \lambda_1)\mu_S^0 + e^0 e^1 \mu_S^p &= 0.\end{aligned}$$

Clearly, $\mu_S^p \neq 0$, as otherwise the whole eigenvector μ_S would be zero. Since the coefficient of μ_S^1 in the first equation is positive, $\mu_S^1/\mu_S^p > 0$. Substituting for μ_S^1 from the first into the second equation and using the inequality $\lambda_1 < -(e^1 - e^0)$ together with the inequality $e^0 > e^1 v^1$ and $e^1 > \frac{1}{2}$ (Lemma D.2 (ii) and (i)),

$$\begin{aligned}\frac{1}{e^1}(w + e^1 - p - \lambda_1)(w - \lambda_1)\frac{\mu_S^0}{\mu_S^p} &= (e^1 - pe^0)v^1 - e^0(w + e^1 - p - \lambda_1)\\ &= e^1 v^1 - e^0(w + e^1 - pe^1 - \lambda_1)\\ &= e^1 v^1 - e^0(r + e^0 + e^1 - \lambda_1)\\ &< e^0 - e^0(r + 2e^1) < 0.\end{aligned}$$

In conclusion, $\mu_S^0/\mu_S^p < 0$. □

Proof of Lemma C.6. We will only prove that $\dot{e}_{S,t}^1 > 0$ here. The proof that $\dot{e}_{S,t}^0 < 0$ is lengthy and is deferred to Section D.1. More specifically, it follows from Lemma D.7(i).

Consider a solution $(e_{S,t}^1, e_{S,t}^0, p_{S,t})$ of Problem 1 with $\hat{p} \in [0, p_{S,*})$ such that $\dot{p}_{S,t} > 0$ for all $t \geq 0$. By Lemma C.3, the variables converge to their critical point values. By Lemma C.5, the direction in which the solution converges to the critical point satisfies $\mu_S^1/\mu_S^p > 0$. This implies that $\dot{e}_{S,t}^1 > 0$ for t large.

Suppose, for contradiction, that $\dot{e}_{S,t}^1 > 0$ fails at some $t_0 > 0$. Then, by continuity, there exists $t_1 \in [t_0, \infty)$ such that $\dot{e}_{S,t_1}^1 = 0$. Let t_1 be the greatest time with this property. Then, $\ddot{e}_{S,t_1}^1 \geq 0$, since $0 = \dot{e}_{S,t_1}^1 < \dot{e}_{S,t}^1$ for all $t > t_1$.

Recall that the dynamics of $e_{S,t}^1$ are governed by the ODE

$$\dot{e}_{S,t}^1 = \tfrac{1}{2}(e_{S,t}^1)^2 - (r + p_{S,t} e_{S,t}^1)(1 - e_{S,t}^1)$$

(see equation (27)). Taking the time derivative and evaluating at $t = t_1$, after noting that $\dot{e}_{S,t_1}^1 = 0$, we obtain

$$\ddot{e}_{S,t_1}^1 = -e_{S,t_1}^1(1 - e_{S,t_1}^1)\dot{p}_{S,t_1} < 0,$$

since $e_{S,t}^1 \in (0, 1)$ and $\dot{p}_{S,t} > 0$ for all $t > 0$ by assumption. This is a contradiction. □

Proof of Lemma C.4. Since the trajectory satisfies Problem 1 for $t > 0$, the time-shifted trajectory starting at any $t_0 > 0$ is a solution of Problem 1 with $\hat{p} = p_{S,t_0}$. By Lemma C.3, it converges to the critical point; by Lemma C.6, $\dot{e}^1_{S,t} > 0$ and $\dot{e}^0_{S,t} < 0$ for all $t > 0$.

Condition $\dot{p}_{S,0} > 0$: Since $e^1_{S,t}$ is increasing and $e^0_{S,t}$ is decreasing for $t > 0$, with $p_{S,t}$ strictly increasing toward $p_{S,*}$, we have $e^1_{S,0} \le e^1_{S,*}$, $e^0_{S,0} \ge e^0_{S,*}$, and $p_{S,0} < p_{S,*}$. By (29),

$$\dot{p}_{S,0} = (1 - p_{S,0})(e^0_{S,0} - p_{S,0}e^1_{S,0}) > (1 - p_{S,0})(e^0_{S,*} - p_{S,*}e^1_{S,*}) = 0.$$

Condition $e^1_{S,0} > 0$: Since $e^1_{S,t}$ is increasing and converges to $e^1_{S,*} < 1$, we have $e^1_{S,0} < 1$. Suppose $e^1_{S,0} = 0$. Then by (27), $\dot{e}^1_{S,0} = -r < 0$, contradicting the continuity of $\dot{e}^1_{S,t} > 0$ for $t > 0$.

Condition $e^0_{S,0} \in (0, 1 - e^1_{S,0})$: Since $e^0_{S,t}$ is decreasing and converges to $e^0_{S,*} > 0$, we have $e^0_{S,0} > 0$. It remains to show that $e^0_{S,0} < 1 - e^1_{S,0}$. Since $e^0_{S,t} + e^1_{S,t} < 1$ for all $t > 0$, by continuity $e^0_{S,0} + e^1_{S,0} \le 1$. Suppose for contradiction that $e^0_{S,0} + e^1_{S,0} = 1$. Then, by (27) and (28),

$$\dot{e}^0_{S,0} + \dot{e}^1_{S,0} = \tfrac{1}{2}(e^0_{S,0})^2 + (r + p_{S,0}e^1_{S,0})[e^0_{S,0} - (1 - e^1_{S,0})] = \tfrac{1}{2}(e^0_{S,0})^2 > 0,$$

which contradicts $e^0_{S,t} + e^1_{S,t} < 1$ for all $t > 0$. Therefore, $e^0_{S,0} < 1 - e^1_{S,0}$.

Condition $p_{S,0} \in [0, 1)$: By continuity, $p_{S,0} \ge 0$, and since $p_{S,t}$ increases toward $p_{S,*} < 1$, we have $p_{S,0} < 1$. □

D.1 Monotonicity of Effort $e^0_{S,t}$ (Proof of Lemma C.6)

Consider $\hat{p} \in [0, p_{S,*})$ such that Problem 1 with initial condition $p_{S,0} = \hat{p}$ has a unique solution $(e^0_{S,t}, e^1_{S,t}, p_{S,t})$. Let $E^1_S(p)$ and $E^0_S(p)$ be the functions verifying the Markov property of the solution, as specified in Problem 1. Assume also that $\dot{p}_{S,t} > 0$ for all $t \ge 0$.

The difficulty of showing that $\dot{e}^0_{S,t} < 0$ for all $t \ge 0$ is that its proof relies on the fact that $\dot{e}^1_{S,t} < e^0_{S,t}\dot{p}_{S,t}$ for every $t \ge 0$. This is equivalent to showing that $\dot{e}^1_{S,t}/\dot{p}_{S,t} = (E^1_S)'(p_{S,t}) < e^0_{S,t}$.

The function $E^1_S(p)$ solves the ODE

$$X'(p) = \nu(X(p), p) := \frac{F^1_S(X(p), E^0_S(p), p)}{F^p_S(X(p), E^0_S(p), p)}, \tag{7}$$

where the functions F^1_S, F^p_S are defined in (27)-(29). The function ν allows us to analyze the ODE for E^1_S while regarding E^0_S as given.

To show that $(E^1_S)'(p) \le m$ for a given $m \ge 0$, we analyze the isoclines $Y(p; m)$ of the

ODE (7) – functions that map p to x such that x solves the equation $\nu(x,p) = m$. The analysis proceeds as follows, based on specific lemmas:

- Lemma D.4 establishes key identities at the critical point for future use.
- Lemma D.5 shows that if the isocline $Y(p;m)$ is well defined and if it has slope less than m, then so does $E_S^1(p)$.
- Lemma D.6 establishes conditions under which the isocline $Y(p;m)$ has slope less than m.
- Lemma D.7 verifies the conditions of Lemma D.6 for $m = e_{S,*}^0$, and establishes that $\dot{e}_{S,t}^0 < 0$ and $\dot{e}_{S,t}^1 < e_{S,*}^0 \dot{p}_{S,t}$ for any $t \geq 0$.

D.2 Identities at the Critical Point

The following lemma establishes identities at the critical point that are essential to the proof of Lemma D.7.

Lemma D.4. *At the critical point* $(e^1, e^0, p) = (e_{S,*}^1, e_{S,*}^0, p_{S,*})$, *the following relationships hold:*

(i) $(e^1)^2 - e^1 + e^0\,(2e^1 + r + e^0 - p) = rp(2 - e^1)$;

(ii) $2r - re^1 + 2e^0 - 2e^0e^1 - 3(e^0)^2 = r(2e^0 + e^1)$.

Proof. (i) We analyze the left-hand side of (i) by examining the terms $(e^1)^2 - e^1$, $2e^0e^1$, and $e^0\,(r + e^0 - p)$ separately. Note that we use the fact that $e^0 = pe^1$ and $e^1 = e^0/p$ in the proof.

The term $(e^1)^2 - e^1$ can be expressed as $(1 - 1/e^1) \cdot (e^1)^2$. Substituting for $(e^1)^2$ using equation (3), we have

$$(e^1)^2 - e^1 = \left(1 - 1/e^1\right) \cdot e^0(2r + 3e^0) = (e^0 - p)(2r + 3e^0).$$

The term $2e^0e^1$ can be represented as $2p \cdot (e^1)^2$. Substituting for $(e^1)^2$ using equation (2), we obtain

$$2e^0e^1 = 2p \cdot 2(r + e^0)(1 - e^1) = -(e^0 - p)(4r + 4e^0).$$

Next we use the obtained identities to substitute into the left-hand side of (i), and obtain

$$\begin{aligned}
&\left[(e^1)^2 - e^1\right] + \left(2e^0e^1\right) + e^0\left(r + e^0 - p\right) \\
&\quad = (e^0 - p)(2r + 3e^0) - (e^0 - p)(4r + 4e^0) + e^0\left(r + e^0 - p\right) \\
&\quad = r(2p - e^0) \\
&\quad = rp(2 - e^1).
\end{aligned}$$

(ii) Rewriting the left-hand side of (ii) using equations (2) and (3), we obtain

$$\begin{aligned}
&re^1 + 2(r + e^0)(1 - e^1) - 3(e^0)^2 \\
&= re^1 + (e^1)^2 - 3(e^0)^2 \\
&= re^1 + 2re^0.
\end{aligned}$$

□

D.3 Isocline Analysis

Consider any $m \geq 0$. To implicitly characterize the isoclines of the ODE $X'(p) = \nu(X(p), p)$, we define the function α as follows

$$\alpha(x, p; m) := F_S^1\big(x, E_S^0(p), p\big) - m \cdot F_S^p\big(x, E_S^0(p), p\big), \quad \forall\, (x, p) \in \mathbb{R}_+ \times [\hat{p}, p_{S,*}].$$

Note that $\alpha(x, p; m) = \big(\nu(x, p) - m\big) F_S^p\big(x, E_S^0(p), p\big)$.

Substituting the expressions for F_S^1 and F_S^p from equations (27) and (29) into the definition of $\alpha(x, p; m)$, we have

$$\begin{aligned}
\alpha(x, p; m) &= F_S^1\big(x, E_S^0(p), p\big) - mF_S^p\big(x, E_S^0(p), p\big) \\
&= \big[\tfrac{1}{2}x^2 - (r + px)(1 - x)\big] - m\big[(1 - p)(E_S^0(p) - px)\big].
\end{aligned}$$

Collecting terms in x, we obtain

$$\alpha(x, p; m) = \big(\tfrac{1}{2} + p\big)\, x^2 + \big(r - p + p(1 - p)m\big)x - \big(r + (1 - p)E_S^0(p)m\big). \tag{8}$$

Lemma D.5. *For any $p \in [\hat{p}, p_{S,*}]$ and $m \geq 0$, the function $\alpha(\cdot, p; m)$ possesses a unique positive root, denoted $Y(p; m)$, and $\alpha_x(Y(p; m), p; m) > 0$. Additionally, given $\tilde{p} \in [\hat{p}, p_{S,*})$, if*

$$Y'(p; m) < m \;\; \textit{for all} \;\; p \in (\tilde{p}, p_{S,*})$$

then

$$(E_S^1)'(p) < m \;\; \textit{for all} \;\; p \in [\tilde{p}, p_{S,*}).$$

Proof. As $x \mapsto \alpha(x, p; m)$ is a quadratic function with a positive leading coefficient and a negative intercept, it has a unique positive root, $Y(p; m)$. At this root, $\alpha_x(x, p; m) > 0$. Additionally, for any $x \geq 0$,

$$\alpha(x, p; m) > 0 \quad \Longleftrightarrow \quad x > Y(p; m). \tag{9}$$

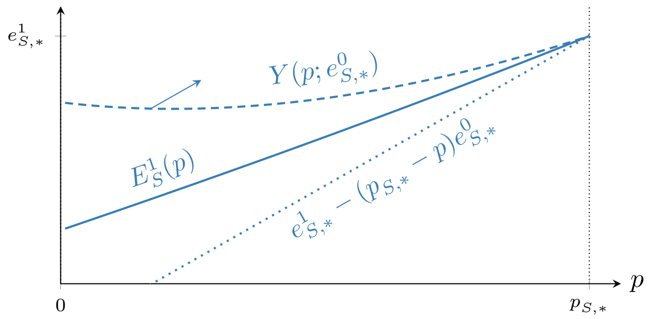


Figure 3: Graphical representation of the function $E^1_S(p)$ (solid curve) in relation to the isocline $Y(p;e^0_{S,*})$ (dashed curve). Both curves lie above the line with slope $e^0_{S,*}$ passing through the point $(p_{S,*}, e^1_{S,*})$ (dotted line). An arrow marks the directed field of the ODE $X'(p) = \nu(X(p), p)$ at a chosen point on the isocline $Y(p;e^0_{S,*})$.

***Claim* 1.** *For any $m \geq 0$, $Y(p_{S,*};m) = e^1_{S,*}$.*

Proof of Claim 1. Given $F^1_S(e^1_{S,*}, e^0_{S,*}, p_{S,*}) = F^p_S(e^1_{S,*}, e^0_{S,*}, p_{S,*}) = 0$, $\alpha(e^1_{S,*}, p_{S,*}; m) = 0$. Thus, $e^1_{S,*}$ is the unique root of $x \mapsto \alpha(x, p_{S,*}; m)$. ■

***Claim* 2.** *For any $p \in [\hat{p}, p_{S,*})$ and $m \geq 0$,*

$$(E^1_S)'(p) \geq m \quad \iff \quad E^1_S(p) \geq Y(p;m),$$

with equality in one implying equality in the other.

Proof of Claim 2. Since $(E^1_S)'(p) = \nu(E^1_S(p), p)$ and

$$\alpha(x,p;m) = \big(\nu(x,p) - m\big) \cdot F^p_S(x, E^0_S(p), p),$$

with $F^p_S(E^1_S(p), E^0_S(p), p) > 0$,[4] we have $(E^1_S)'(p) \geq m \iff \alpha(E^1_S(p), p; m) \geq 0$. By (9), this is equivalent to $E^1_S(p) \geq Y(p;m)$. Equalities match by definition. ■

To conclude, suppose for contradiction that for some $\tilde{p} \in [\hat{p}, p_{S,*})$, we have $Y'(p;m) < m$ for all $p \in (\tilde{p}, p_{S,*})$, but $(E^1_S)'(p_0) \geq m$ at some $p_0 \in [\tilde{p}, p_{S,*})$. Since $E^1_S(p_{S,*}) = Y(p_{S,*}, m)$ (by Claim 1), the functions $E^1_S(p)$ and $Y(p;m)$ intersect in the interval $(p_0, p_{S,*}]$. Let p_1 denote the leftmost intersection point, which exists by continuity of the functions and their differing slopes at p_0. This implies $E^1_S(p) > Y(p;m)$ for every $p \in (p_0, p_1)$, and consequently, $(E^1_S)'(p) > m$ throughout this interval (by Claim 2). This contradicts the mean value theorem applied to $p \mapsto E^1_S(p) - Y(p;m)$, since $E^1_S(p_0) \geq Y(p_0;m)$ (by Claim 2), $E^1_S(p_1) = Y(p_1;m)$, and $Y'(p;m) < m$ for all $p \in (p_0, p_1)$. □

[4] The conditions of Problem 1 require $\dot{p}_{S,t} > 0$.

D.4 Slope of the Isoclines

To characterize the slope of the isocline $Y(p;m)$ implicitly, we introduce the function β as

$$\beta(x,p;m) = \alpha_p(x,p;m) + m\alpha_x(x,p;m),$$

for all $(x,p) \in \mathbb{R}_+ \times [0, p_{S,*}]$. Incorporating the partial derivatives, we get

$$\beta(x,p;m) = x^2 + (2m-1)x + m\big[r - p + m\,p(1-p) + E_S^0(p) - (1-p)(E_S^0)'(p)\big], \quad (10)$$

indicating that $\beta(\cdot,p;m)$ is a quadratic polynomial in x.

Lemma D.6. *For a given $\tilde{p} \in [\hat{p}, p_{S,*})$ and $m \geq 0$ satisfying $(1-p_{S,*})m \geq \frac{1}{2} - e^1_{S,*}$, if*

$$\beta\big(e^1_{S,*} - (p_{S,*} - p)m, p; m\big) > 0 \quad \textit{for all} \quad p \in [\tilde{p}, p_{S,*}], \quad (11)$$

then $(E^1_S)'(p) < m$ for all $p \in [\tilde{p}, p_{S,*})$.

Proof. Consider any $p \in [\hat{p}, p_{S,*}]$ and $m \geq 0$.

***Claim* 1.** *$\beta(Y(p;m),p;m) > 0$ if and only if $Y'(p;m) < m$.*

Proof of Claim 1. Given that $Y(p;m)$ is the root of $\alpha(\cdot,p;m)$, by the Implicit Function Theorem,

$$Y'(p;m) = -\frac{\alpha_p(Y(p;m),p;m)}{\alpha_x(Y(p;m),p;m)},$$

where the denominator is strictly positive as per Lemma D.5. This yields

$$0 = \alpha_p(Y(p;m),p;m) + Y'(p;m) \cdot \alpha_x(Y(p;m),p;m),$$

which, when compared to the definition of β, verifies the claim. ■

***Claim* 2.** *If $\beta(\hat{x},p;m) > 0$ for some $\hat{x} \in \left[\frac{1}{2} - m, Y(p;m)\right]$, then $Y'(p;m) < m$.*

Proof of Claim 2. By (10), $\beta(\cdot,p;m)$ is a quadratic function with minimum at $x = \frac{1}{2} - m$. Thus, $\beta(\hat{x},p;m) > 0$ implies $\beta(Y(p;m),p;m) > 0$, which yields $Y'(p;m) < m$ by Claim 1. ■

Fix $\tilde{p} \in [\hat{p}, p_{S,*})$ and $m \geq 0$ such that $(1-p_{S,*})m \geq \frac{1}{2} - e^1_{S,*}$, and define $\hat{Y}(p;m) := e^1_{S,*} - (p_{S,*} - p)m$ for all $p \in [\hat{p}, p_{S,*}]$.

***Claim* 3.** *If both $\beta(\hat{Y}(p;m),p;m) > 0$ and $\hat{Y}(p;m) \leq Y(p;m)$ hold, then $Y'(p;m) < m$.*

Proof of Claim 3. The condition $(1-p_{S,*})m \geq \frac{1}{2} - e^1_{S,*}$ implies $\hat{Y}(0;m) \geq \frac{1}{2} - m$. Thus, $\hat{Y}(p;m) \geq \frac{1}{2} - m$. The claim then follows from Claim 2. ■

***Claim* 4.** *If $\beta\big(\hat{Y}(p;m),p;m\big) > 0$ for all $p \in [\tilde{p}, p_{S,*}]$, then $Y'(p;m) < m$ for all $p \in (\tilde{p}, p_{S,*})$.*

Proof of Claim 4. Suppose otherwise that $Y'(\underline{p};m) \geq m$ for some $\underline{p} \in [\tilde{p}, p_{S,*})$, while $\beta\big(\hat{Y}(p;m),p;m\big) > 0$ for all $p \in [\tilde{p}, p_{S,*}]$. By Claim 3, this implies $\hat{Y}(\underline{p};m) > Y(\underline{p};m)$. Since $\hat{Y}(p_{S,*};m) = e^1_{S,*} = Y(p_{S,*};m)$ (by Claim 1 in the proof of Lemma D.5) and $Y'(p_{S,*};m) < m$ (by Claim 3), the functions $Y(\cdot;m)$ and $\hat{Y}(\cdot;m)$ must intersect in the interval $(\underline{p}, p_{S,*})$. Let $\overline{p}$ be the largest point of intersection. At this point, $Y(p;m)$ must intersect $\hat{Y}(p;m)$ from below. However, by Claim 3, we have $Y'(\overline{p};m) < m$, which is a contradiction. ■

By combining Lemma D.5 and Claim 4, we arrive at the desired conclusion. □

D.5 Proof of Lemma D.7

Lemma D.7. *Let $\hat{p} < p_{S,*}$ be such that Problem 1 with initial condition $p_{S,0} = \hat{p}$ has a unique solution $(e^0_{S,t}, e^1_{S,t}, p_{S,t})$. Assume that $\dot{p}_{S,t} > 0$ for all $t \geq 0$. Then for any $t \geq 0$:*

(i) $\dot{e}^0_{S,t} < 0$;

(ii) $\dot{e}^1_{S,t} < e^0_{S,*}\dot{p}_{S,t}$.

Proof. Define $p^{(i)}$ as the infimum of values in $[\hat{p}, p_{S,*}]$ such that inequality (i) holds for all $p \in (p^{(i)}, p_{S,*})$, and similarly for $p^{(ii)}$. We show that $p^{(i)} = p^{(ii)} = \hat{p}$.

By inequality (i), integrating over $(\tilde{p}, p_{S,*})$ yields $E^0_S(\tilde{p}) > e^0_{S,*}$ for all $\tilde{p} \in [p^{(i)}, p_{S,*})$.

***Claim* 1.** *Either $p^{(i)} = \hat{p}$ or $p^{(i)} < p^{(ii)}$.*

Proof of Claim 1. By Lemma C.5, $(E^0_S)'(p_{S,*}) = m^0_* < 0$. By continuous differentiability of $E^0_S(p)$, inequality (i) holds in a neighborhood of $p_{S,*}$, so $p^{(i)} < p_{S,*}$.

If $p^{(i)} = \hat{p}$, the claim holds trivially. If $p^{(i)} \in (0, p_{S,*})$, then since $\dot{p}_t > 0$ whenever $p_t < p_{S,*}$, condition (i) at $p = p_t$ is equivalent to $\dot{e}^0_{S,t} < 0$. Fix $t_0 \geq 0$ such that $p_{t_0} = p^{(i)}$. By the definition of $p^{(i)}$, $\dot{e}^0_{S,t_0} = 0$ and $\ddot{e}^0_{S,t_0} \leq 0$. However, taking the time derivative of (28) and substituting $\dot{e}^0_{S,t_0} = 0$, we obtain

$$\begin{aligned}\ddot{e}^0_{S,t_0} &= e^1_{S,t_0}(-\dot{e}^1_{S,t_0} + e^0_{S,t_0}\dot{p}_{t_0}) + p_{t_0}e^0_{S,t_0}\dot{e}^1_{S,t_0} \\ &> \dot{p}_{t_0}e^1_{S,t_0}[E^0_S(p_{t_0}) - (E^1_S)'(p_{t_0})] \\ &> \dot{p}_{t_0}e^1_{S,t_0}[e^0_{S,*} - (E^1_S)'(p^{(i)})].\end{aligned}$$

For $\ddot{e}^0_{S,t_0} \leq 0$ to hold, we must have $(E^1_S)'(p^{(i)}) > e^0_{S,*}$. By continuity of $(E^1_S)'(p)$, we conclude that $p^{(ii)} > p^{(i)}$. ■

Claim 2. $p^{(ii)} \leq p^{(i)}$.

Proof of Claim 2. We apply Lemma D.6 with $m = e^0_{S,*}$ and $\tilde{p} = p^{(i)}$. To verify the lemma's conditions, note first that the inequality $(1-p_{S,*})m \geq \frac{1}{2} - e^1_{S,*}$ holds since $e^1_{S,*} > \frac{1}{2}$ (by Lemma D.2 (i)), so that

$$(1-p_{S,*})e^0_{S,*} > 0 > \tfrac{1}{2} - e^1_{S,*}.$$

To verify (11), we substitute $x := e^1_{S,*} - (p_{S,*} - p)m$ in (10) and separate the $(p_{S,*} - p)$ terms

$$\begin{aligned}
&\beta\big(e^1_{S,*} - (p_{S,*} - p)e^0_{S,*}, p; e^0_{S,*}\big) \\
&\quad = \beta\big(e^1_{S,*}, p_{S,*}; e^0_{S,*}\big) + (p_{S,*} - p)e^0_{S,*}\left[(p_{S,*} - p)e^0_{S,*} - 2e^1_{S,*} + (1 - 2e^0_{S,*})\right] \\
&\qquad + e^0_{S,*}\left[-p + e^0_{S,*}p(1-p) + E^0_S(p) - (1-p)(E^0_S)'(p)\right] \\
&\qquad - e^0_{S,*}\left[-p_{S,*} + e^0_{S,*}p_{S,*}(1-p_{S,*}) + e^0_{S,*} - (1-p_{S,*})m^0_*\right], \\
&\quad = \beta\big(e^1_{S,*}, p_{S,*}; e^0_{S,*}\big) + (p_{S,*} - p)e^0_{S,*}\left[2p_{S,*}e^0_{S,*} + 2 - 2e^1_{S,*} - 3e^0_{S,*}\right] \\
&\qquad + e^0_{S,*}\left[E^0_S(p) - e^0_{S,*}\right] + e^0_{S,*}\left[(1-p_{S,*})m^0_* - (1-p)(E^0_S)'(p)\right].
\end{aligned}$$

By inequality (i), $\beta\big(e^1_{S,*} - (p_{S,*} - p)e^0_{S,*}, p; e^0_{S,*}\big)$ is bounded below by

$$\beta\big(e^1_{S,*}, p_{S,*}; e^0_{S,*}\big) + (p_{S,*} - p)e^0_{S,*}\left(2p_{S,*}e^0_{S,*} + 2 - 2e^1_{S,*} - 3e^0_{S,*}\right) + (1-p_{S,*})e^0_{S,*}m^0_* =: \hat{\beta}(p),$$

for any $p \geq p^{(i)}$. Since $\hat{\beta}(p)$ is linear in p, verifying (11) reduces to verifying that $\hat{\beta}(p_{S,*}) > 0$ and $\hat{\beta}(0) > 0$. We have

$$\begin{aligned}
\hat{\beta}(p_{S,*}) &= \beta\big(e^1_{S,*}, p_{S,*}; m\big) + (1-p_{S,*})e^0_{S,*}m^0_* \\
&= (e^1_{S,*})^2 + (2e^0_{S,*} - 1)e^1_{S,*} + e^0_{S,*}\left[r - p_{S,*} + p_{S,*}(1-p_{S,*})e^0_{S,*} + e^0_{S,*}\right] \\
&> (e^1_{S,*})^2 - e^1_{S,*} + e^0_{S,*}\left(2e^1_{S,*} + r + e^0_{S,*} - p_{S,*}\right) = rp_{S,*}(2 - e^1_{S,*}) > 0,
\end{aligned}$$

where the last identity is by Lemma D.4 (i); and likewise

$$\begin{aligned}
\hat{\beta}(0) &= \hat{\beta}(p_{S,*}) + p_{S,*}e^0_{S,*}\left(2p_{S,*}e^0_{S,*} + 2 - 2e^1_{S,*} - 3e^0_{S,*}\right) \\
&> p_{S,*}\left[2r - re^1_{S,*} + e^0_{S,*}\left(2p_{S,*}e^0_{S,*} + 2 - 2e^1_{S,*} - 3e^0_{S,*}\right)\right] \\
&> p_{S,*}\left[2r - re^1_{S,*} + 2e^0_{S,*} - 2e^0_{S,*}e^1_{S,*} - 3(e^0_{S,*})^2\right] = rp_{S,*}(2e^0_{S,*} + e^1_{S,*}) > 0,
\end{aligned}$$

where the last identity is by Lemma D.4 (ii). ■

By Claim 1 and Claim 2, $p^{(i)} = p^{(ii)} = \hat{p}$. Thus, inequalities (i) and (ii) hold for the entire interval $(\hat{p}, p_{S,*})$. □

E Supplementary Appendix: Proofs for Section 6 (Patent Race with One Firm Known to Have a Breakthrough)

We use the normalization $v = 1$ and $a = 1$.

Rather than characterizing the trajectory of the vector $(v^{1,A}_{A,t}, v^{1,B}_{A,t}, v^{0,B}_{A,t}, p^{B}_{A,t})$ via the ODEs (7)–(10), with optimal efforts substituted from (11), we equivalently characterize the trajectory of the vector $(e^{1,A}_{A,t}, e^{1,B}_{A,t}, e^{0,B}_{A,t}, p^{B}_{A,t})$ as follows:

Problem 1. A trajectory $(e^{1,A}_{A,t}, e^{1,B}_{A,t}, e^{0,B}_{A,t}, p^{B}_{A,t})$ satisfies the following system of ODEs:

$$\dot{e}^{1,A}_{A,t} = \tfrac{1}{2}(e^{1,A}_{A,t})^2 - (r + p^{B}_{A,t}e^{1,B}_{A,t})(1 - e^{1,A}_{A,t}) \quad =: F^{1,A}_{A}(e^{1,A}_{A,t}, e^{1,B}_{A,t}, e^{0,B}_{A,t}, p^{B}_{A,t}) \tag{12}$$

$$\dot{e}^{1,B}_{A,t} = \tfrac{1}{2}(e^{1,B}_{A,t})^2 - (r + e^{1,A}_{A,t})(1 - e^{1,B}_{A,t}) \quad =: F^{1,B}_{A}(e^{1,A}_{A,t}, e^{1,B}_{A,t}, e^{0,B}_{A,t}, p^{B}_{A,t}) \tag{13}$$

$$\dot{e}^{0,B}_{A,t} = \tfrac{1}{2}(e^{0,B}_{A,t})^2 - \tfrac{1}{2}(e^{1,B}_{A,t})^2 + (r + e^{1,A}_{A,t})e^{0,B}_{A,t} \quad =: F^{0,B}_{A}(e^{1,A}_{A,t}, e^{1,B}_{A,t}, e^{0,B}_{A,t}, p^{B}_{A,t}) \tag{14}$$

$$\dot{p}^{B}_{A,t} = (1 - p^{B}_{A,t})\big(e^{0,B}_{A,t} - p^{B}_{A,t}e^{1,B}_{A,t}\big) \quad =: F^{p}_{A}(e^{1,A}_{A,t}, e^{1,B}_{A,t}, e^{0,B}_{A,t}, p^{B}_{A,t}), \tag{15}$$

with the initial condition $p^{B}_{A,0} = \hat{p} \in [0,1)$, constraints $e^{1,A}_{A,t}, e^{1,B}_{A,t} \in (0,1)$, $e^{0,B}_{A,t} \in (0, 1 - e^{1,B}_{A,t})$, and $p^{B}_{A,t} \in [0,1)$ for all $t \geq 0$. Assume strategies are Markov in the posterior belief $p^{B}_{A,t}$: there exist functions $E^{1,A}_{A}$, $E^{1,B}_{A}$, and $E^{0,B}_{A}$ such that $e^{1,A}_{A,t} = E^{1,A}_{A}(p^{B}_{A,t})$, $e^{1,B}_{A,t} = E^{1,B}_{A}(p^{B}_{A,t})$, and $e^{0,B}_{A,t} = E^{0,B}_{A}(p^{B}_{A,t})$ for all $t \geq 0$.

The proof of Proposition 4 has two steps. We first establish local existence and uniqueness of a solution in a neighborhood of the critical point. We then extend this solution uniquely backward in time until the posterior belief reaches 0. The first step relies on the following three lemmas.

Lemma E.1. *The system of ODEs* (12)–(15) *has a unique critical point* $(e^{1,A}_{A,*}, e^{1,B}_{A,*}, e^{0,B}_{A,*}, p^{B}_{A,*})$ *such that* $e^{1,A}_{A,*}, e^{1,B}_{A,*} \in (0,1)$, $e^{0,B}_{A,*} \in (0, e^{1,B}_{A,*})$, *and* $p^{B}_{A,*} \in (0,1)$.

Lemma E.2. *The Jacobian at the critical point of the system of ODEs* (12)–(15) *has one negative eigenvalue and three eigenvalues with a positive real part.*

Lemma E.3. *Any solution of Problem 1 converges to the critical point* $(e^{1,A}_{A,*}, e^{1,B}_{A,*}, e^{0,B}_{A,*}, p^{B}_{A,*})$ *as* $t \to \infty$. *Moreover, if* $p^{B}_{A,0} < p^{B}_{A,*}$, *then the belief* $p^{B}_{A,t}$ *is increasing over time.*

The proof of Lemma E.3 relies on the following auxiliary lemma.

Lemma E.4. *Define functions* $\phi^{1,A}, \phi^{1,B} : [0,1] \to [0,1]$ *as follows. For* $p^B \in [0,1]$, *let* $\phi^{1,B}(p^B)$ *denote the unique fixed point of the function* $\Upsilon(z) = \psi^{-1}(\psi^{-1}(p^B z))$, *and set* $\phi^{1,A}(p^B) := \psi(\phi^{1,B}(p^B))$. *Then the functions* $\phi^{1,A}$ *and* $\phi^{1,B}$ *are well defined and strictly increasing. Moreover:*

(i) If $p^B_{A,t} \geq \underline{p}^B$ *for all* $t \geq t_0$*, then* $e^{1,A}_{A,t} \geq \phi^{1,A}(\underline{p}^B)$ *and* $e^{1,B}_{A,t} \geq \phi^{1,B}(\underline{p}^B)$ *for all* $t \geq t_0$*.*

(ii) If $p^B_{A,t} \leq \overline{p}^B$ *for all* $t \geq t_0$*, then* $e^{1,A}_{A,t} \leq \phi^{1,A}(\overline{p}^B)$ *and* $e^{1,B}_{A,t} \leq \phi^{1,B}(\overline{p}^B)$ *for all* $t \geq t_0$*.*

E.1 Proofs of Propositions 4 (Existence and Uniqueness) and 5 (Effort)

The structure of the proof of Proposition 4 is analogous to that of Proposition 2, but it differs significantly in certain technical aspects. Specifically, while the former relies on the fact that $(E^0_S)'(p) < 0$, here we use a weaker analog: $(E^{0,B}_A)'(p) < 0$ whenever $E^{0,B}_A(p) - pE^{1,B}_A(p)$ is close to zero. This is easier to prove analytically than $(E^{0,B}_A)'(p) < 0$ globally (which can be verified numerically).

Before proving Proposition 4, we establish some properties of a candidate solution. Suppose that there exist functions $E^{1,A}_A(p)$, $E^{1,B}_A(p)$, $E^{0,B}_A(p)$ representing a solution of the system of ODEs (12)–(14) on some interval $[p_L, p^B_{A,*}]$, where $0 < p_L < p^B_{A,*}$. Note that $\sigma^p_A(p) > 0$ for all $p \in [p_L, p^B_{A,*})$: by Lemma E.3, the belief $p^B_{A,t}$ is monotone increasing, so $\dot{p}^B_{A,t} = \sigma^p_A(p^B_{A,t}) > 0$ whenever $p^B_{A,t} < p^B_{A,*}$. Denote

$$
\begin{aligned}
\sigma^{1,A}_A(p) &:= \tfrac{1}{2}[E^{1,A}_A(p)]^2 - [r + pE^{1,B}_A(p)]\,[1 - E^{1,A}_A(p)] \\
\sigma^{1,B}_A(p) &:= \tfrac{1}{2}[E^{1,B}_A(p)]^2 - [r + E^{1,A}_A(p)]\,[1 - E^{1,B}_A(p)] \\
\sigma^{0,B}_A(p) &:= \tfrac{1}{2}[E^{0,B}_A(p)]^2 - \tfrac{1}{2}[E^{1,B}_A(p)]^2 + [r + E^{1,A}_A(p)]E^{0,B}_A(p) \\
\sigma^{p}_A(p) &:= (1 - p)[E^{0,B}_A(p) - pE^{1,B}_A(p)]
\end{aligned}
$$

corresponding to the right-hand sides of (12)–(15). As established above, $\sigma^p_A(p) > 0$ for all $p \in [p_L, p^B_{A,*})$. For notational convenience, set $e^{1,A}_{A,t} := e^{1,A}(p^B_{A,t})$, $e^{1,B}_{A,t} := e^{1,B}(p^B_{A,t})$, $e^{0,B}_{A,t} := e^{0,B}(p^B_{A,t})$, where $p^B_{A,t}$ is the solution of the ODE

$$
\dot{p}^B_{A,t} = \sigma^p_A(p^B_{A,t}) = (1 - p^B_{A,t})\left[e^{0,B}(p^B_{A,t}) - p^B_{A,t}e^{1,B}(p^B_{A,t})\right]
$$

with initial condition $p^B_{A,0} = p_L$. Then the ODEs (12)–(14) can be rewritten as

$$
(E^{1,A}_A)'(p) = \frac{\sigma^{1,A}_A(p)}{\sigma^p_A(p)}, \qquad (E^{1,B}_A)'(p) = \frac{\sigma^{1,B}_A(p)}{\sigma^p_A(p)}, \qquad (E^{0,B}_A)'(p) = \frac{\sigma^{0,B}_A(p)}{\sigma^p_A(p)}. \tag{16}
$$

We obtain the following two lemmas.

Lemma E.5. $(E^{1,A}_A)'(p) > 0$ *and* $(E^{1,B}_A)'(p) > 0$ *for all* $p \in [p_L, p^B_{A,*})$*.*

Lemma E.6. *For any* $\delta > 0$ *there exists* $\varepsilon > 0$ *such that* $\left|E^{0,B}_A(p) - pE^{1,B}_A(p)\right| \leq \varepsilon$ *implies* $\sigma^{0,B}_A(p) < 0$ *for all* $p \in [p_L, p^B_{A,*} - \delta]$*.*

The backward extension of the solution additionally relies on one more lemma.

Lemma E.7. *Let* $(e^{1,A}_{A,t}, e^{1,B}_{A,t}, e^{0,B}_{A,t}, p^B_{A,t})$ *be a solution of Problem 1 for* $t > 0$*. Then the constraints of Problem 1 and* $\dot{p}^B_{A,0} > 0$ *also hold at* $t = 0$*.*

Proof of Proposition 4. Let p_L be the infimum of the $\mathcal{P}$ values of $\hat{p} \in [0, p^B_{A,*})$ such that there exists a unique solution $(e^{1,A}_{A,t}, e^{1,B}_{A,t}, e^{0,B}_{A,t}, p^B_{A,t})$ to Problem 1 with initial condition $p^B_{A,0} = \hat{p}$. We need to prove that $p_L = 0$. We will do this by contradiction. Suppose that $p_L > 0$.

We begin by showing local existence of unique solution. At the critical point $(e^{1,A}_{A,*}, e^{1,B}_{A,*}, e^{0,B}_{A,*}, p^B_{A,*})$, the Jacobian has one negative eigenvalue and three eigenvalues with a positive real part (Lemma E.2); accordingly, there is a unique stable direction $(\mu^{1,A}_A, \mu^{1,B}_A, \mu^{0,B}_A, \mu^{p,B}_A)$. Moreover, $\mu^{p,B}_A \neq 0$ (Lemma E.9). The Hartman–Grobman theorem (Teschl, 2012, Theorem 9.9) therefore guarantees the existence of a locally unique trajectory $(e^{1,A}_{A,t}, e^{1,B}_{A,t}, e^{0,B}_{A,t}, p^B_{A,t})$ of (12)–(15) with $\dot{p}^B_{A,t} > 0$ for all $t > 0$ that converges to the critical point as $t \to \infty$. Moreover, since $e^{1,A}_{A,*}, e^{1,B}_{A,*} \in (0,1)$, $e^{0,B}_{A,*} \in (0, 1 - e^{1,B}_{A,*})$, and $p^B_{A,*} \in (0,1)$, the constraints on the trajectory are satisfied for all sufficiently large t. The Markov property is also trivially satisfied, as $p^B_{A,t}$ is strictly increasing. Taking into account that every solution to Problem 1 converges to the critical point (Lemma E.3), there must exist a unique solution to Problem 1 for some $\hat{p} \in [0, p^B_{A,*})$. We therefore conclude that $p_L < p^B_{A,*}$.

Let $(\bar{e}^{1,A}_A, \bar{e}^{1,B}_A, \bar{e}^{0,B}_A, \bar{p}^B_A)$ be the limit of $(e^{1,A}_{A,0}, e^{1,B}_{A,0}, e^{0,B}_{A,0}, p^B_{A,0})$ corresponding to the solution of Problem 1 with the initial condition $\hat{p}$ as $\hat{p} \searrow p_L$. Since the system of ODEs (12)–(15) is autonomous, solutions to Problem 1 with progressively decreasing initial condition $\hat{p}$ are extensions of each other except for that the time shifts, and so the limit is well defined.

The vector function $(F^{1,A}_A, F^{1,B}_A, F^{0,B}_A, F^p_A)$ is locally Lipschitz continuous, so by the Picard–Lindelöf theorem (see, e.g., Teschl, 2012, Theorem 2.2), there exists a unique solution $(e^{1,A}_{A,t}, e^{1,B}_{A,t}, e^{0,B}_{A,t}, p^B_{A,t})$ to the system of ODEs (12)–(15) with the initial conditions $(e^{1,A}_{A,0}, e^{1,B}_{A,0}, e^{0,B}_{A,0}, p^B_{A,0}) = (\bar{e}^{1,A}_A, \bar{e}^{1,B}_A, \bar{e}^{0,B}_A, \bar{p}^B_A)$ for t on some neighborhood of 0. In fact, the system has a unique solution for all $t \geq 0$, as those correspond to the solution of Problem 1 with initial condition $\hat{p} > p_L$. What is more, the existence of a unique solution for $t < 0$ on some neighborhood of 0 means that by shifting time we can conclude that Problem 1 has a unique solution for some $\hat{p} < p_L$ so long as we ensure that Problem 1 constraints are satisfied. By Lemma E.7, the constraints indeed hold at $t = 0$, including $\dot{p}^B_{A,0} > 0$. Since the solution of the system of ODEs (12)–(15) is an analytic function, the constraints are also satisfied on some neighborhood of 0. As $\dot{p}^B_{A,0} > 0$, $p^B_{A,t} < p_L$ for $t < 0$. We conclude that there exists some $\hat{p} < p_L$ such that Problem 1 with initial condition $p^B_{A,0} = \hat{p}$ has a unique solution, contradicting $p_L > 0$. □

Proof of Proposition 5. The existence of a solution is guaranteed by Proposition 4, the claim then follows directly from Lemma E.5. □

E.2 Proofs of Propositions 6–7 (Comparisons)

Proof of Proposition 6. We first establish the inequality between efforts, $e^{1,A}_{A,t} < e^{1,B}_{A,t}$. The corresponding statement for continuation values, $v^{1,A}_{A,t} > v^{1,B}_{A,t}$, follows from the identities in (11).

We begin by verifying that the effort inequality holds at the critical point, i.e., $e^{1,A}_{A,*} < e^{1,B}_{A,*}$. Recall from Lemma E.1 that the critical point efforts satisfy $p^B_{A,*}e^{1,B}_{A,*} = \psi(e^{1,A}_{A,*})$ and $e^{1,A}_{A,*} = \psi(e^{1,B}_{A,*})$, where $\psi(z) = z^2/(2(1-z)) - r$ is the function analyzed in Lemma D.1.

Suppose, for the sake of contradiction, that $e^{1,B}_{A,*} \le e^{1,A}_{A,*}$. Then $p^B_{A,*}e^{1,B}_{A,*} \le p^B_{A,*}e^{1,A}_{A,*} < e^{1,A}_{A,*}$, where the strict inequality follows from $p^B_{A,*} < 1$. Since ψ^{-1} is strictly increasing, applying it to both sides yields $\psi^{-1}(p^B_{A,*}e^{1,B}_{A,*}) < \psi^{-1}(e^{1,A}_{A,*})$. By the critical point conditions, $e^{1,A}_{A,*} = \psi(e^{1,B}_{A,*})$ and $p^B_{A,*}e^{1,B}_{A,*} = \psi(e^{1,A}_{A,*})$, so $e^{1,A}_{A,*} < e^{1,B}_{A,*}$, which contradicts the initial assumption.

Next, suppose to the contrary that $e^{1,B}_{A,t} > e^{1,A}_{A,t}$ does not hold for all $t \ge 0$. Let $\tau \ge 0$ be the smallest real number such that $e^{1,B}_{A,t} > e^{1,A}_{A,t}$ holds for all $t > \tau$. Then necessarily $e^{1,B}_{A,\tau} = e^{1,A}_{A,\tau}$ and $\dot{e}^{1,B}_{A,\tau} \ge \dot{e}^{1,A}_{A,\tau}$. However, by (12)–(13),

$$\begin{aligned}
\dot{e}^{1,A}_{A,\tau} &= \tfrac{1}{2}(e^{1,A}_{\tau})^2 - (r + p^B_{A,\tau}e^{1,B}_{\tau})(1 - e^{1,A}_{\tau}) \\
&= \tfrac{1}{2}(e^{1,B}_{\tau})^2 - (r + p^B_{A,\tau}e^{1,A}_{\tau})(1 - e^{1,B}_{\tau}) \\
&> \tfrac{1}{2}(e^{1,B}_{\tau})^2 - (r + e^{1,A}_{\tau})(1 - e^{1,B}_{\tau}) = \dot{e}^{1,B}_{A,\tau},
\end{aligned}$$

where the strict inequality uses $p^B_{A,\tau} < 1$ and $e^{1,B}_{\tau} = e^{1,A}_{\tau}$. This is a contradiction.

It remains to prove the comparison of continuation values from the perspective of an informed party. Since firm B is informed, its continuation value is $v^{1,B}_{A,t}$. Let $v^{1,A/B}_t$ denote the continuation value of firm A from the perspective of an informed party. These continuation values satisfy the following system of ODEs:

$$\begin{aligned}
-\dot{v}^{1,A/B}_t &= \tfrac{1}{2}(e^{1,A}_{A,t})^2 - (r + e^{1,B}_{A,t})v^{1,A/B}_t, \\
-\dot{v}^{1,B}_{A,t} &= \tfrac{1}{2}(e^{1,B}_{A,t})^2 - (r + e^{1,A}_{A,t})v^{1,B}_{A,t}.
\end{aligned}$$

The latter ODE is identical to (8). The former ODE differs from (7) for $v^{1,A}_{A,t}$ by the absence of the $p^B_{A,t}$ term, since the informed party knows firm B's state. Thus, the perceived hazard rate of B's patenting is simply its true hazard rate, $e^{1,B}_{A,t}$.

Consider the critical point. From the inequality $e^{1,A}_{A,*} < e^{1,B}_{A,*}$ established above, it follows that

$$v_*^{1,A/B} = \frac{\frac{1}{2}(e_{A,*}^{1,A})^2}{r + e_{A,*}^{1,B}} < \frac{\frac{1}{2}(e_{A,*}^{1,B})^2}{r + e_{A,*}^{1,A}} = v_{A,*}^{1,B}.$$

Therefore, $v_t^{1,A/B} < v_{A,t}^{1,B}$ necessarily holds for t large. Suppose, to the contrary, that this inequality does not hold for all $t \geq 0$. Let $\tau \geq 0$ be the smallest real number such that $v_t^{1,A/B} < v_{A,t}^{1,B}$ holds for all $t > \tau$. Then necessarily $v_\tau^{1,A/B} = v_\tau^{1,B}$ and $\dot{v}_\tau^{1,A/B} \leq \dot{v}_{A,\tau}^{1,B}$. Together with $e_\tau^{1,A} < e_\tau^{1,B}$, this implies

$$\dot{v}_\tau^{1,A/B} = -\tfrac{1}{2}(e_\tau^{1,A})^2 + (r + e_\tau^{1,B})v_\tau^{1,A/B} > -\tfrac{1}{2}(e_\tau^{1,B})^2 + (r + e_\tau^{1,A})v_\tau^{1,B} = \dot{v}_{A,\tau}^{1,B},$$

which is a contradiction. □

Proof of Proposition 7. Throughout this proof, we omit the "$*$" subscript on variables denoting critical point values. Recall that, in the private information case, the critical point values satisfy equations (4)–(5), with $p = e^0/e^1$.

$$1 = \left(\frac{e^0}{e^1}\right)^2 + \frac{e^0}{1 - e^1}.$$

The proof proceeds in four steps: *Step 1.* We show that $e^1 < e^{1,B}$. Recall that $p = e^0/e^1$, $e^0 = \psi(e^1)$, and thus equation (5) can be written as

$$1 = \eta(p, e^1) := p^2 + p \cdot \frac{e^1}{1 - e^1},$$

where η is strictly increasing in both arguments. Moreover, from (4),

$$p = \frac{e^0}{e^1} = \frac{\psi(e^1)}{e^1} = \frac{e^1}{2(1 - e^1)} - \frac{r}{e^1},$$

which is strictly increasing in e^1. Therefore, the function $z \mapsto \eta(\psi(z)/z, z)$ is strictly increasing.

Similarly, in the one-sided information case, $p^B = e^{0,B}/e^{1,B}$, $e^{0,B} = \psi(e^{1,A})$, and equation (17) can be written as $1 = \eta(p^B, e^{1,B})$. By Lemma E.8(i), $e^{1,A} < e^{1,B}$. Therefore,

$$\eta\left(\frac{\psi(e^1)}{e^1}, e^1\right) = 1 = \eta\left(\frac{\psi(e^{1,A})}{e^{1,B}}, e^{1,B}\right) < \eta\left(\frac{\psi(e^{1,B})}{e^{1,B}}, e^{1,B}\right),$$

where the strict inequality follows from the monotonicity of η and ψ. Thus, $e^1 < e^{1,B}$.

Step 2. We show that $e^{0,B} < e^0$. Equation (5) can be equivalently written as

$$1 = \theta(e^0, e^1) := \left(\frac{e^0}{e^1}\right)^2 + \frac{e^0}{1 - e^1},$$

where θ is increasing in e^0. For any $z \in [e^1, 1)$,

$$\frac{\partial}{\partial z}\theta(e^0, z) = -\frac{2(e^0)^2}{z^3} + \frac{e^0}{(1-z)^2} > 0,$$

because

$$\frac{z^3}{2(1-z)^2} = [\psi(z) + r] \cdot \frac{z}{1-z} > e^0,$$

where we use $z > e^1 > \frac{1}{2}$ and $\psi(z) \geq \psi(e^1) = e^0$. Therefore, the inequality $e^{1,B} > e^1$ from Step 1 implies $\theta(e^0, e^{1,B}) > \theta(e^0, e^1) = 1$. However, equation (17) gives $\theta(e^{0,B}, e^{1,B}) = 1$, so necessarily $e^{0,B} < e^0$.

Step 3. We show that $e^{1,A} < e^1$. Since ψ is strictly increasing and $e^{0,B} < e^0$ by Step 2, it follows that

$$e^{1,A} = \psi^{-1}(e^{0,B}) < \psi^{-1}(e^0) = e^1.$$

Step 4. We show that $p^B < p$. From Step 1, $e^1 < e^{1,B}$, and since η is strictly increasing in both arguments, it follows from $\eta(p, e^1) = 1 = \eta(p^B, e^{1,B})$ that $p^B < p$. This completes the proof. □

E.3 Proofs of Supporting Lemmas E.1–E.7

Proof of Lemma E.1. We proceed analogously to the proof of Lemma C.1. A critical point of the system (12)–(15) is characterized by $(\dot{e}^{1,A}_{A,t}, \dot{e}^{1,B}_{A,t}, \dot{e}^{0,B}_{A,t}, \dot{p}^B_{A,t}) = (0,0,0,0)$. Dropping the subscript, we obtain $e^{0,B} = p^B e^{1,B}$ and

$$\begin{aligned}
0 &= \tfrac{1}{2}(e^{1,A})^2 - (r + e^{0,B})(1 - e^{1,A}),\\
0 &= \tfrac{1}{2}(e^{1,B})^2 - (r + e^{1,A})(1 - e^{1,B}),\\
0 &= \tfrac{1}{2}(e^{0,B})^2 - \tfrac{1}{2}(e^{1,B})^2 + (r + e^{1,A})e^{0,B}.
\end{aligned}$$

This system is equivalent to

$$
\begin{aligned}
e^{0,B} &= \frac{(e^{1,A})^2}{2(1-e^{1,A})} - r, \\
e^{1,A} &= \frac{(e^{1,B})^2}{2(1-e^{1,B})} - r, \\
1 &= \left(\frac{e^{0,B}}{e^{1,B}}\right)^2 + \frac{e^{0,B}}{1-e^{1,B}}.
\end{aligned} \tag{17}
$$

Define the function $\Gamma(z)$ on $[0, 1)$ by

$$
\Gamma(z) := \left[\frac{\psi(\psi(z))}{z}\right]^2 + \frac{\psi(\psi(z))}{1-z},
$$

where $\psi(z)$ is as introduced in Lemma D.1. A quadruple $(e^{1,A}, e^{1,B}, e^{0,B}, p^B)$ is a critical point of the ODE system (12)–(15) if and only if $e^{0,B} = \psi(e^{1,A})$, $e^{1,A} = \psi(e^{1,B})$, $1 = \Gamma(e^{1,B})$, and $p^B = e^{0,B}/e^{1,B}$.

Consider $z \in [\psi^{-1}(\psi^{-1}(0)), 1)$. The functions $\psi(z)/[1 - \psi(z)]$ and $\psi(z)/z$ are strictly increasing and positive on this domain. Therefore, their product, as well as

$$
\frac{1}{2} \cdot \frac{\psi(z)}{1-\psi(z)} \cdot \frac{\psi(z)}{z} - \frac{r}{z} = \left[\frac{\psi(z)^2}{2(1-\psi(z))} - r\right]\frac{1}{z} = \frac{\psi(\psi(z))}{z},
$$

are strictly increasing and positive. Consequently, $\Gamma(z)$, being the sum of two strictly increasing functions, is strictly increasing. Moreover, $\Gamma(\psi^{-1}(\psi^{-1}(0))) = 0$, and

$$
\Gamma(e^{11}_C) > \left[\frac{\psi(\psi(e^{11}_C))}{e^{11}_C}\right]^2 = \left(\frac{e^{11}_C}{e^{11}_C}\right)^2 = 1,
$$

where e^{11}_C is the equilibrium effort in state 11 under complete information, satisfying $\psi(e^{11}_C) = e^{11}_C$ (see Lemma C.1). Therefore, $e^{1,B}_{A,*} = \Gamma^{-1}(1) \in (\psi^{-1}(\psi^{-1}(0)), e^{11}_C)$ is unique.

By Lemma D.1, $e^{1,B} < e^{11}_C$ implies $e^{1,A} = \psi(e^{1,B}) < e^{1,B}$, which in turn implies $e^{0,B} = \psi(e^{1,A}) < e^{1,A}$. Thus, $p^B = e^{0,B}/e^{1,B} < 1$. Hence, the quadruple $(e^{1,A}, e^{1,B}, e^{0,B}, p^B)$ is the unique critical point of the ODE system (12)–(15).

Finally, the condition $e^{0,B} < 1 - e^{1,B}$ follows from (17). □

Before presenting the proof of Lemma E.2, we establish several useful inequalities at the critical point, summarized in Lemma E.8 below.

Lemma E.8. *At the unique critical point* $(e^{1,A}, e^{1,B}, e^{0,B}, p^B) = (e^{1,A}_{A,*}, e^{1,B}_{A,*}, e^{0,B}_{A,*}, p^B_{A,*})$, *the following hold:*

(i) $v^{1,B} < v^{1,A} < \frac{1}{2} < e^{1,A} < e^{1,B}$*;*

(ii) $e^{0,B} > e^{1,B} v^{1,B}$;

(iii) $r + e^{1,A} + e^{0,B} > e^{1,B}$.

Proof. (i) The inequality $e^{1,A} < e^{1,B}$ was established in the proof of Lemma E.1. Since $v^{1,A} = 1 - e^{1,A}$ and $v^{1,B} = 1 - e^{1,B}$, it remains to show that $\frac{1}{2} < e^{1,A}$. Recall that $1 = \Gamma(e^{1,B})$, $e^{1,A} = \psi(e^{1,B})$, and both ψ and Γ are strictly increasing. Thus, the claim is equivalent to $\frac{1}{2} < \psi(\Gamma^{-1}(1))$. Let $\hat{z} := \psi^{-1}(\frac{1}{2})$. We now show that $\Gamma(\hat{z}) < 1$.

Observe that

$$\tfrac{1}{2} = \psi(\hat{z}) = \tfrac{1}{2} \cdot \frac{\hat{z}^2}{1 - \hat{z}} - r, \quad \text{so} \quad 1 - \hat{z} = \frac{\hat{z}^2}{1 + 2r},$$

and

$$\psi(\psi(\hat{z})) = \psi(\tfrac{1}{2}) = \tfrac{1}{4} - r.$$

We then obtain

$$\begin{aligned}\Gamma(\hat{z}) &= \frac{(\frac{1}{4} - r)^2}{\hat{z}^2} + \frac{\frac{1}{4} - r}{1 - \hat{z}} = \frac{(\frac{1}{4} - r)^2}{\hat{z}^2} + \frac{(\frac{1}{4} - r)(1 + 2r)}{\hat{z}^2} \\ &= \frac{\frac{5}{16} - r - r^2}{\hat{z}^2} < \frac{\frac{5}{16}}{\hat{z}^2} < \frac{5}{8(3 - \sqrt{5})} < 1,\end{aligned}$$

where the last inequality follows from $1 < \hat{z}^2/(1 - \hat{z})$, which implies $\hat{z}^2 > \frac{1}{2}(3 - \sqrt{5})$.

(ii) Suppose, for contradiction, that $e^{0,B} \leq e^{1,B} v^{1,B} = e^{1,B}(1 - e^{1,B})$. Then, from equation (17):

$$1 = \left(\frac{e^{0,B}}{e^{1,B}}\right)^2 + \frac{e^{0,B}}{1 - e^{1,B}} \leq (1 - e^{1,B})^2 + e^{1,B} < (1 - e^{1,B}) + e^{1,B} = 1,$$

a contradiction.

(iii) Recall that $e^{1,A} = \psi(e^{1,B})$, and so

$$r + e^{1,A} = \frac{(e^{1,B})^2}{2(1 - e^{1,B})} > (e^{1,B})^2 > e^{1,B} - e^{0,B},$$

where the first inequality follows from $e^{1,B} > \frac{1}{2}$ (by (i)), and the second from $e^{0,B} > e^{1,B}(1 - e^{1,B})$ (by (ii)). □

Proof of Lemma E.2. Consider the critical point $(e^{1,A}, e^{1,B}, e^{0,B}, p^B) = (e^{1,A}_{A,*}, e^{1,B}_{A,*}, e^{0,B}_{A,*}, p^B_{A,*})$ and define

$$w_A := r + e^{1,A} + e^{0,B}, \quad \text{and} \quad h^B_A := e^{1,B} - e^{0,B}.$$

The Jacobian matrix of the system of ODEs (12)–(15) at the critical point is

$$J_A = \begin{bmatrix} w_A & -p^B v^{1,A} & 0 & -e^{1,B} v^{1,A} \\ -v^{1,B} & w_A + h_A^B & 0 & 0 \\ e^{0,B} & -e^{1,B} & w_A & 0 \\ 0 & -p^B(1-p^B) & 1-p^B & -e^{1,B}(1-p^B) \end{bmatrix}.$$

The eigenvalues of J_A are the complex roots of the characteristic polynomial $Q_A(\lambda) := \det(J_A - \lambda I)$. We compute

$$Q_A(\lambda) = \det \begin{bmatrix} w_A - \lambda & -p^B v^{1,A} & 0 & -e^{1,B} v^{1,A} \\ -v^{1,B} & w_A + h_A^B - \lambda & 0 & 0 \\ e^{0,B} & -e^{1,B} & w_A - \lambda & 0 \\ 0 & -p^B(1-p^B) & 1-p^B & -e^{1,B}(1-p^B) - \lambda \end{bmatrix}.$$

Subtracting $(p^B/e^{1,B})$ times the last column from the second column, and using $e^{1,B}(1 - p^B) = h_A^B$, we obtain

$$Q_A(\lambda) = \det \begin{bmatrix} w_A - \lambda & 0 & 0 & -e^{1,B} v^{1,A} \\ -v^{1,B} & w_A + h_A^B - \lambda & 0 & 0 \\ e^{0,B} & -e^{1,B} & w_A - \lambda & 0 \\ 0 & \frac{p^B}{e^{1,B}}\lambda & 1-p^B & -h_A^B - \lambda \end{bmatrix}.$$

Expanding along the first row yields:

$$\begin{aligned} Q_A(\lambda) &= (w_A - \lambda)^2 (w_A + h_A^B - \lambda)(-h_A^B - \lambda) \\ &\quad + e^{1,B} v^{1,A} \cdot \det \begin{bmatrix} -v^{1,B} & w_A + h_A^B - \lambda & 0 \\ e^{0,B} & -e^{1,B} & w_A - \lambda \\ 0 & \frac{p^B}{e^{1,B}}\lambda & 1-p^B \end{bmatrix} \\ &= (w_A - \lambda)^2 (w_A + h_A^B - \lambda)(-h_A^B - \lambda) \\ &\quad + v^{1,A} v^{1,B} [e^{1,B} h_A^B + p^B (w_A - \lambda)\lambda] - v^{1,A} e^{0,B} h_A^B (w_A + h_A^B - \lambda). \end{aligned}$$

We can express the polynomial in terms of its coefficients as $Q_A(\lambda) = \lambda^4 - d_3\lambda^3 + d_2\lambda^2 - d_1\lambda + d_0$. Taking into account the inequalities $w_A = r + e^{1,A} + e^{0,B} > e^{1,A}$, $w_A + h_A^B = r + e^{1,A} + e^{1,B} > e^{1,B}$, and $v^{1,B} < v^{1,A} < \frac{1}{2} < e^{1,A} < e^{1,B}$ (by Lemma E.8(i)), we have

$$
\begin{aligned}
d_0/h_A^B = Q_A(0)/h_A^B &= -(w_A)^2(w_A + h_A^B) + v^{1,A}v^{1,B}e^{1,B} - v^{1,A}e^{0,B}(w_A + h_A^B) \\
&< -(w_A)^2(w_A + h_A^B) + v^{1,A}v^{1,B}e^{1,B} \\
&< -(w_A)^2(w_A + h_A^B) + (w_A)^2(w_A + h_A^B) = 0.
\end{aligned}
$$

Thus, Q_A has at least one positive root and at least one negative root (recall that $Q_A(\lambda)$ diverges to infinity as $\lambda \to \infty$ or $\lambda \to -\infty$).

We now show that Q_A has a unique negative root, denoted $\lambda_{A,1}$. Observe that

$$
d_3 = 2w_A + (w_A + h_A^B) - h_A^B = 3w_A > 0.
$$

Since $Q_A'''(\lambda) = 24\lambda - 6d_3 < 0$ for all $\lambda \leq 0$, the second derivative $Q_A''(\lambda)$ is strictly decreasing on $(-\infty, 0]$. Moreover,

$$
\frac{Q_A''(0)}{2} = d_2 = 3(w_A)^2 - (h_A^B)^2 - h_A^B w_A - v^{1,A}v^{1,B}p^B > 0,
$$

where the inequality follows from $w_A > v^{1,A} > v^{1,B}$ (by Lemma E.8(i)) and $w_A > h_A^B$ (by Lemma E.8(iii)). It follows that $Q_A''(\lambda) > 0$ for all $\lambda \leq 0$, so Q_A is strictly convex on $(-\infty, 0]$. Consequently, Q_A has a unique negative root.

It remains to show that Q_A has no root with a non-positive real part and a nonzero imaginary part. Suppose, for contradiction, that $\lambda_{A,2}$ is such a root. Then its complex conjugate $\lambda_{A,3}$ must also be a root. Let $\lambda_{A,4}$ denote the unique positive root of Q_A. By *Vieta's formulas*, we have

$$
d_3 = \lambda_{A,1} + \lambda_{A,2} + \lambda_{A,3} + \lambda_{A,4}.
$$

Since all roots except $\lambda_{A,4}$ have non-positive real parts, it must be that $\lambda_{A,4} > d_3 = 3w_A > 0$. But this would imply $Q_A(3w_A) < 0$. However,

$$
\begin{aligned}
Q_A(3w_A) &= (2w_A)^2(2w_A - h_A^B)(3w_A + h_A^B) \\
&\quad + v^{1,A}v^{1,B}\left[e^{1,B}h_A^B - 6p^B(w_A)^2\right] + v^{1,A}e^{0,B}h_A^B(2w_A - h_A^B) \\
&> (2w_A)^2 w_A(3w_A) + (w_A)^2[-6(w_A)^2] = 6(w_A)^4 > 0,
\end{aligned}
$$

where the inequality follows from $w_A > v^{1,A} > v^{1,B}$ (by Lemma E.8(i)) and $w_A > h_A^B$ (by Lemma E.8(iii)). This contradiction completes the proof. □

Lemma E.9. *The eigenvector* $\mu_A = (\mu_A^{1,A}, \mu_A^{1,B}, \mu_A^{0,B}, \mu_A^{p,B})$ *of the Jacobian matrix* J_A *associated with the negative eigenvalue* $\lambda_{A,1}$ *satisfies* $\mu_A^{1,A}/\mu_A^{p,B} > 0$, $\mu_A^{1,B}/\mu_A^{p,B} > 0$, *and* $\mu_A^{0,B}/\mu_A^{p,B} < 0$.

Proof. The eigenvector μ_A is characterized by the vector equation $(J_A - \lambda_{A,1} I)\mu_A = 0$, which yields

$$
\begin{aligned}
(w_A - \lambda_{A,1})\mu_A^{1,A} - p^B v^{1,A}\mu_A^{1,B} - e^{1,B}v^{1,A}\mu_A^{p,B} &= 0,\\
-v^{1,B}\mu_A^{1,A} + (w_A + h_A^B - \lambda_{A,1})\mu_A^{1,B} &= 0,\\
e^{0,B}\mu_A^{1,A} - e^{1,B}\mu_A^{1,B} + (w_A - \lambda_{A,1})\mu_A^{0,B} &= 0.
\end{aligned}
$$

Substituting for $\mu_A^{1,A}$ from the second equation into the others, we obtain

$$
\begin{aligned}
\Big[(w_A - \lambda_{A,1})(w_A + h_A^B - \lambda_{A,1}) - p^B v^{1,A} v^{1,B}\Big]\mu_A^{1,B} - e^{1,B}v^{1,A}v^{1,B}\mu_A^{p,B} &= 0,\\
\Big[e^{0,B}(w_A + h_A^B - \lambda_{A,1}) - e^{1,B}v^{1,B}\Big]\mu_A^{1,B} + (w_A - \lambda_{A,1})v^{1,B}\mu_A^{0,B} &= 0.
\end{aligned}
$$

Since $w_A > e^{1,A} > \frac{1}{2} > v^{1,A} > v^{1,B}$, we have $(w_A)^2 > v^{1,A}v^{1,B}$, so the coefficient of $\mu_A^{1,B}$ in the first equation is positive. Consequently, $\mu_A^{1,B}/\mu_A^{p,B} > 0$, and thus also $\mu_A^{1,A}/\mu_A^{p,B} > 0$ (since $\mu_A^{p,B} \neq 0$, as otherwise the entire vector μ_A would be zero). Finally, the coefficient of $\mu_A^{1,B}$ in the second equation is positive because $e^{0,B} > e^{1,B}v^{1,B}$ (by Lemma E.8(ii)), and $w_A + h_A^B - \lambda_{A,1} > w_A + h_A^B = r + e^{1,A} + e^{1,B} > 1$ (by Lemma E.8(i)). Therefore, $\mu_A^{0,B}/\mu_A^{p,B} < 0$. □

Proof of Lemma E.4. We first establish that the functions $\phi^{1,A}$ and $\phi^{1,B}$ are well defined. Recall that the function $\psi(z)$ is defined in Lemma D.1, which also states that its inverse $\psi^{-1} : [-r, \infty) \to [0, 1)$ is continuous, strictly increasing, and strictly concave. Moreover, for any $p^B \in (0, 1]$, the function $z \mapsto \Upsilon(z) = \psi^{-1}(\psi^{-1}(p^B z))$ is also strictly concave, because composing an increasing strictly concave map with an affine transformation preserves strict concavity. Additionally, $\Upsilon(0) > 0$ and $\Upsilon(1) < 1$, so Υ has a unique fixed point on $[0, 1]$.

For any $p^B \in [0, 1]$, the values $e^{1,A} = \phi^{1,A}(p^B)$ and $e^{1,B} = \phi^{1,B}(p^B)$ satisfy $p^B e^{1,B} = \psi(e^{1,A})$ and $e^{1,A} = \psi(e^{1,B})$. These equations correspond to the critical point conditions for (12) and (13):

$$
\begin{aligned}
0 &= \tfrac{1}{2}(e^{1,A})^2 - (r + p^B e^{1,B})(1 - e^{1,A}),\\
0 &= \tfrac{1}{2}(e^{1,B})^2 - (r + e^{1,A})(1 - e^{1,B}).
\end{aligned}
$$

That is, if $e_{A,t}^{1,A} = \phi^{1,A}(p_{A,t}^B)$ and $e_{A,t}^{1,B} = \phi^{1,B}(p_{A,t}^B)$, then $\dot{e}_{A,t}^{1,A} = 0$ and $\dot{e}_{A,t}^{1,B} = 0$.

The functions $\phi^{1,B}$ and $\phi^{1,A}$ are continuous and satisfy

$$z > \psi^{-1}(\psi^{-1}(p^B z)) \qquad \text{if and only if} \qquad z > \phi^{1,B}(p^B). \tag{18}$$

To show that $\phi^{1,A}$ and $\phi^{1,B}$ are strictly increasing, let $p^B_{A,1}, p^B_{A,2} \in [0,1)$ with $p^B_{A,1} < p^B_{A,2}$. For $z_2 = \phi^{1,B}(p^B_{A,2})$, the monotonicity of ψ^{-1} implies $z_2 = \psi^{-1}(\psi^{-1}(p^B_2 z_2)) > \psi^{-1}(\psi^{-1}(p^B_1 z_2))$. Then (18) gives $\phi^{1,B}(p^B_2) = z_2 > \phi^{1,B}(p^B_1)$. Therefore, $\phi^{1,B}(p^B)$ is strictly increasing, and since $\phi^{1,A}(p^B) = \psi(\phi^{1,B}(p^B))$, $\phi^{1,A}(p^B)$ is also strictly increasing.

We now prove part (i); part (ii) follows analogously. Let $\underline{p}^B \in [0,1]$ and $t_0 \geq 0$ be such that $p^B_{A,t} \geq \underline{p}^B$ for all $t \geq t_0$. Define

$$\underline{e}^{1,A} = \inf_{t \geq t_0} e^{1,A}_{A,t} \qquad \text{and} \qquad \underline{e}^{1,B} = \inf_{t \geq t_0} e^{1,B}_{A,t}.$$

Let

$$\chi(z_1, z_2) := \tfrac{1}{2} z_1^2 - (r + z_2)(1 - z_1).$$

Note that $\chi(z_1, z_2)$ is increasing in z_1 and decreasing in z_2 for $z_1, z_2 \in [0,1)$, and $\chi(z_1, z_2) \geq 0$ if and only if $z_2 \leq \psi(z_1)$.

We claim that

$$0 \leq \chi(\underline{e}^{1,A}, \underline{p}^B \underline{e}^{1,B}) \qquad \text{and} \qquad 0 \leq \chi(\underline{e}^{1,B}, \underline{e}^{1,A}). \tag{19}$$

Suppose the first inequality is violated, i.e., $\chi(\underline{e}^{1,A}, \underline{p}^B \underline{e}^{1,B}) = -2\delta < 0$. Define

$$\mathcal{T} = \{ t \geq t_0 : \chi(e^{1,A}_{A,t}, \underline{p}^B \underline{e}^{1,B}) \leq -\delta \}.$$

By definition of $\underline{e}^{1,A}$, the set $\mathcal{T}$ is nonempty. Since $t \mapsto \chi(e^{1,A}_{A,t}, \underline{p}^B \underline{e}^{1,B})$ is continuous, $\mathcal{T}$ is closed. For all $t \in \mathcal{T}$, we have $\dot{e}^{1,A}_{A,t} = \chi(e^{1,A}_{A,t}, p^B_{A,t} e^{1,B}_{A,t}) \leq \chi(e^{1,A}_{A,t}, \underline{p}^B \underline{e}^{1,B}) \leq -\delta$. Since $\mathcal{T}$ is closed, any boundary point t' of $\mathcal{T}$ belongs to $\mathcal{T}$, so $\dot{e}^{1,A}_{A,t'} \leq -\delta$. This implies $[t', t'+\varepsilon] \subset \mathcal{T}$ for some $\varepsilon > 0$. Therefore, $\mathcal{T} = [t_1, +\infty)$ for some t_1. Consequently, $e^{1,A}_{A,t}$ decreases at rate at least δ for all $t \geq t_1$, contradicting that it is bounded below by $\underline{e}^{1,A}$. The second inequality in (19) follows similarly.

To show $\underline{e}^{1,A} \geq \phi^{1,A}(\underline{p}^B)$ and $\underline{e}^{1,B} \geq \phi^{1,B}(\underline{p}^B)$, note that (19) implies $\underline{p}^B \underline{e}^{1,B} \leq \psi(\underline{e}^{1,A})$ and $\underline{e}^{1,A} \leq \psi(\underline{e}^{1,B})$. Since ψ is increasing, we have $\underline{p}^B \underline{e}^{1,B} \leq \psi(\psi(\underline{e}^{1,B}))$, and thus $\psi^{-1}(\psi^{-1}(\underline{p}^B \underline{e}^{1,B})) \leq \underline{e}^{1,B}$. By (18), this gives $\underline{e}^{1,B} \geq \phi^{1,B}(\underline{p}^B)$. The inequality $\underline{e}^{1,A} \geq \phi^{1,A}(\underline{p}^B)$ follows similarly.

Finally, by definition of $\underline{e}^{1,A}$ and $\underline{e}^{1,B}$, we obtain $e^{1,A}_{A,t} \geq \phi^{1,A}(\underline{p}^B)$ and $e^{1,B}_{A,t} \geq \phi^{1,B}(\underline{p}^B)$ for all $t \geq t_0$. □

Lemma E.10. *The function $\phi^{1,B}(p^B)$ is concave on $[0,1]$.*

Proof of Lemma E.10. Recall that for any $p^B \in [0,1]$, $\phi^{1,B}(p^B)$ is the unique fixed point of the mapping $z \mapsto \Upsilon(z) := \psi^{-1}(\psi^{-1}(p^B z))$ on $[0,1]$. Equivalently, $\phi^{1,B}(p^B)$ is the inverse function of $\Psi(z) := \psi(\psi(z))/z$, defined for $z \in (0,1) \cap \psi^{-1}([0,1)) \cap \psi^{-1}(\psi^{-1}([0,1)))$. Since $\phi^{1,B}(p^B)$ is strictly increasing (see Lemma E.4), its concavity is equivalent to the convexity of Ψ.

We now establish that $\Psi''(z) > 0$. Take any $z \in (0,1)$ such that $\psi(z), \psi(\psi(z)) \in [0,1)$. Then,

$$\begin{aligned}
\Psi'(z) &= \psi'(\psi(z))\psi'(z)z^{-1} - \psi(\psi(z))z^{-2}, \\
\Psi''(z) &= \psi''(\psi(z))[\psi'(z)]^2 z^{-1} + \psi'(\psi(z))\psi''(z)z^{-1} - 2\psi'(\psi(z))\psi'(z)z^{-2} + 2\psi(\psi(z))z^{-3} \\
&\geq \psi''(\psi(z))[\psi'(z)]^2 z^{-1} - 2\psi'(\psi(z))\psi'(z)z^{-2}, \qquad (20)
\end{aligned}$$

where we have used that, by the choice of z, $\psi(\psi(z)) \geq 0$, $\psi''(z) > 0$, and $\psi'(\psi(z)) > 0$.

For $z < 1$, we have $\psi(z) < 1$, and thus

$$\frac{2-z}{1-z} > 2 > [2-\psi(z)]\,[1-\psi(z)].$$

This implies

$$\frac{z^2(2-z)}{2(1-z)^2} > \frac{z^2}{2(1-z)} \cdot [2-\psi(z)]\,[1-\psi(z)],$$

which further yields

$$\frac{z^2(2-z)}{2(1-z)^2} > \left(\frac{z^2}{2(1-z)} - r\right)[2-\psi(z)]\,[1-\psi(z)].$$

This can be rewritten as

$$\psi'(z)z > \psi(z)[2-\psi(z)]\,[1-\psi(z)].$$

Note that

$$\psi'(\psi(z)) = \frac{\psi(z)[2-\psi(z)]}{2[1-\psi(z)]^2} \qquad \text{and} \qquad \psi''(\psi(z)) = \frac{1}{[1-\psi(z)]^3}.$$

Multiplying the previous inequality by $\psi'(z)z^{-2}(1-\psi(z))^{-3}$, we obtain

$$\psi''(\psi(z))[\psi'(z)]^2 z^{-1} > 2\psi'(\psi(z))\psi'(z)z^{-2}.$$

Combining this with (20), we conclude that $\Psi''(z) > 0$, which establishes the convexity of Ψ and thus the concavity of $\phi^{1,B}(p^B)$. $\square$

Lemma E.11. *Assume that $p^B_{A,t} < 1$ for all $t \geq 0$ and $p^B_{A,t} \to p^B_{A,\infty}$, where $p^B_{A,\infty} \in [0,1]$. Then $(e^{1,A}_{A,t}, e^{1,B}_{A,t}, e^{0,B}_{A,t}, p^B_{A,t}) \to (e^{1,A}_{A,*}, e^{1,B}_{A,*}, e^{0,B}_{A,*}, p^B_{A,*})$ as $t \to +\infty$.*

Proof of Lemma E.11. Since $p^B_{A,t} \to p^B_{A,\infty}$, for any $\varepsilon > 0$ there exists T such that $|p^B_{A,t} - p^B_{A,\infty}| < \varepsilon$ for all $t \geq T$. By Lemma E.4, it follows that $e^{1,A}_{A,t} \to e^{1,A}_{A,\infty} := \phi^{1,A}(p^B_{A,\infty})$ and $e^{1,B}_{A,t} \to e^{1,B}_{A,\infty} := \phi^{1,B}(p^B_{A,\infty})$ as $t \to +\infty$.

Consider the ODE (14), which can be written as

$$\dot{e}^{0,B}_{A,t} = \kappa^{0,B,t}_A(e^{0,B}_{A,t}), \quad \text{where} \quad \kappa^{0,B,t}_A(z) := \tfrac{1}{2}z^2 - \tfrac{1}{2}(e^{1,B}_{A,t})^2 + (r + e^{1,A}_{A,t})z.$$

The functions $\kappa^{0,B,t}_A(z)$ are continuously differentiable and converge uniformly to $\kappa^{0,B,\infty}_A(z) := \frac{1}{2}z^2 - \frac{1}{2}(e^{1,B}_{A,\infty})^2 + (r + e^{1,A}_{A,\infty})z$ as $t \to +\infty$. Since $\kappa^{0,B,\infty}_A$ is a quadratic polynomial with positive leading coefficient, $\kappa^{0,B,\infty}_A(0) < 0$, and $\kappa^{0,B,\infty}_A(e^{1,B}_{A,\infty}) > 0$, it admits a unique positive root, which we denote $e^{0,B}_{A,\infty}$. Thus, $e^{0,B}_{A,\infty} \in (0, e^{1,B}_{A,\infty})$ and $(\kappa^{0,B,\infty}_A)'(e^{0,B}_{A,\infty}) > 0$. By Lemma D.3, it follows that $e^{0,B}_{A,t} \to e^{0,B}_{A,\infty}$.

Since $e^{0,B}_{A,\infty} < e^{1,B}_{A,\infty}$, it follows from the ODE (15) that $p^B_{A,\infty} \leq 1$, because otherwise $\dot{p}^B_{A,t}$ would necessarily be negative for large t, precluding $p^B_{A,t}$ from exceeding 1.

Moreover, $p^B_{A,\infty} = 1$ is excluded: it would require $e^{0,B}_{A,\infty} = e^{1,B}_{A,\infty}$ at the critical point, but then $\dot{e}^{0,B}_{A,\infty} = (r + e^{1,A}_{A,\infty})e^{1,B}_{A,\infty} > 0$, contradicting stationarity. Therefore, $(e^{1,A}_{A,\infty}, e^{1,B}_{A,\infty}, e^{0,B}_{A,\infty}, p^B_{A,\infty})$ is a critical point of the system (12)–(15) with $p^B_{A,\infty} < 1$, and thus, by Lemma E.1, we obtain $(e^{1,A}_{A,\infty}, e^{1,B}_{A,\infty}, e^{0,B}_{A,\infty}, p^B_{A,\infty}) = (e^{1,A}_{A,*}, e^{1,B}_{A,*}, e^{0,B}_{A,*}, p^B_{A,*})$. $\square$

Proof of Lemma E.3. In any Markov perfect Bayesian equilibrium, the posterior belief $p^B_{A,t}$ must be monotonic. Otherwise, there would exist $0 < t_1 < t_2$ such that $p^B_{A,t_1} = p^B_{A,t_2}$ but $\dot{p}^B_{A,t_1} \neq \dot{p}^B_{A,t_2}$, which would violate the Markov property.[5]

Since $p^B_{A,t}$ is monotonic and bounded, it must converge. The remainder follows directly from Lemma E.11. $\square$

Proof of Lemma E.5. Suppose that to the contrary at least one of the inequalities

$$(E^{1,A}_A)'(p) > 0 \text{ and } (E^{1,B}_A)'(p) > 0$$

is violated at some $p \in [p_L, p^B_{A,*})$.

[5]Indeed, $p^B_{A,t}$ is the sole state variable in the game. If $p^B_{A,t}$ is identical at two distinct times, then the associated values $e^{1,A}_{A,t}$, $e^{1,B}_{A,t}$, and $e^{0,B}_{A,t}$ must also coincide, which in turn implies that $\dot{p}^B_{A,t}$ must be equal at those times.

Define $\overline{p} = \inf\{p \in (p_L, p^B_{A,*}) : (E^{1,A}_A)'(p) > 0 \text{ and } (E^{1,B}_A)'(p) > 0\}$. Since the direction $(\mu^{1,A}_A, \mu^{1,B}_A, \mu^{0,B}_A, \mu^{p,B}_A)$ in which the solution has to converge to the critical point satisfies $\mu^{1,A}_A/\mu^{p,B}_A > 0$ and $\mu^{1,B}_A/\mu^{p,B}_A > 0$, both of the efforts have positive derivative for p close enough to $p^B_{A,*}$. Hence, $\overline{p} < p^B_{A,*}$. What is more, by the initial assumption of the proof, $\overline{p} > p_L$. Then $(E^{1,A}_A)'(\overline{p}) = 0$ or $(E^{1,B}_A)'(\overline{p}) = 0$, and hence at time $\tau \geq 0$ such that $p^B_{A,\tau} = \overline{p}$ we have $\dot{e}^{1,A}_{A,\tau} = 0$ or $\dot{e}^{1,B}_{A,\tau} = 0$. We treat these two cases individually.

Case 1. Let $\dot{e}^{1,A}_{A,\tau} = 0$. Taking the derivative of (12) and using that $\dot{e}^{1,A}_{A,\tau} = 0$, we obtain

$$\ddot{e}^{1,A}_{A,\tau} = -(\dot{p}^B_{A,\tau} e^{1,B}_\tau + p^B_{A,\tau}\dot{e}^{1,B}_{A,\tau})(1 - e^{1,A}_\tau) < 0.$$

However, this is in contradiction with the fact that $(E^{1,A}_A)'(\overline{p}) = 0$, while $(E^{1,A}_A)'(p) > 0$ for $p \in (\overline{p}, p^B_{A,*})$.

Case 2. Let $\dot{e}^{1,B}_{A,\tau} = 0$. As in Case 1, taking the derivative of (13) and using that $\dot{e}^{1,B}_{A,\tau} = 0$, we obtain

$$\ddot{e}^{1,B}_{A,\tau} = -\dot{e}^{1,A}_{A,\tau}(1 - e^{1,B}_\tau) < 0,$$

which is in contradiction with the fact that $(E^{1,B}_A)'(\overline{p}) = 0$, while $(E^{1,B}_A)'(p) > 0$ for $p \in (\overline{p}, p^B_{A,*})$. □

Proof of Lemma E.6. Fix $\delta > 0$. Temporarily fix $\varepsilon > 0$ and let $p \in [p_L, p^B_{A,*} - \delta]$ satisfy $\left| E^{0,B}_A(p) - pE^{1,B}_A(p) \right| < \varepsilon$.

Consider the function

$$\xi^{0,B}(z) := \tfrac{1}{2}z^2 - \tfrac{1}{2}[E^{1,B}_A(p)]^2 + [r + E^{1,A}_A(p)]z.$$

Its derivative with respect to z is $r + z + E^{1,A}_A(p)$, which is in absolute value lower than $r + 2$, when $z < 1$. Thus,

$$\left|\xi^{0,B}\big(E^{0,B}_A(p)\big) - \xi^{0,B}\big(pE^{1,B}_A(p)\big)\right| < (r+2)\left|E^{0,B}_A(p) - pE^{1,B}_A(p)\right| < (r+2)\varepsilon$$

Therefore, in order to prove that $\dot{e}^{0,B}(p) < 0$, which is equivalent to $\xi^{0,B}\big(E^{0,B}_A(p)\big) < 0$, it is sufficient to show that $\xi^{0,B}\big(pE^{1,B}_A(p)\big) < -(r+2)\varepsilon$. This can be rewritten as

$$-\tfrac{1}{2}(1-p^2)[E^{1,B}_A(p)]^2 + [r + E^{1,A}_A(p)]pE^{1,B}_A(p) < -(r+2)\varepsilon. \tag{21}$$

If we regard the left-hand side of (21) as a quadratic function of $E^{1,B}_A(p)$, it has a negative leading coefficient, positive linear coefficient, and zero intercept. Thus, it is a decreasing function whenever its value is negative. By Lemma E.5, $E^{1,A}_A(p) \leq e^{1,A}_{A,*}$ and by Lemma E.4, $\phi^{1,B}(p) \leq E_A^{1,B}(p)$, whenever $p \leq p_{A,*}^B$. Hence, to prove (21) it is enough to show that

$$C := -\tfrac{1}{2}(1-p^2)[\phi^{1,B}(p)]^2 + \left(r + e_{A,*}^{1,A}\right) p\phi^{1,B}(p) < -(r+2)\varepsilon. \tag{22}$$

For $p \leq p_{A,*}^B$ we have

$$C \leq \phi^{1,B}(p) \left[-\tfrac{1}{2}\left(1 - (p_{A,*}^B)^2\right)\phi^{1,B}(p) + p\left(r + e_{A,*}^{1,A}\right)\right].$$

By Lemma E.4, $\phi^{1,B}(p)$ is increasing and by Lemma E.10 it is concave on $[0, p_{A,*}^B]$. The former implies that $\phi^{1,B}(p) \geq \phi^{1,B}(0)$, whereas the latter implies that

$$\phi^{1,B}(p) \geq \frac{p_{A,*}^B - p}{p_{A,*}^B}\phi^{1,B}(0) + \frac{p}{p_{A,*}^B}\phi^{1,B}(p_{A,*}^B) = \frac{p_{A,*}^B - p}{p_{A,*}^B}\phi^{1,B}(0) + \frac{p}{p_{A,*}^B}e_{A,*}^{1,B}.$$

Note that by its definition $\phi^{1,B}(0) = \psi^{-1}(\psi^{-1}(0)) > 0$. What is more, since $\sigma_A^{0,B}(p_{A,*}^B) = 0$ and $e^{0,B}(p_{A,*}^B) = p_{A,*}^B e^{1,B}(p_{A,*}^B)$, at the critical point the left-hand side of the inequality (21) is equal to 0, which gives us that

$$\tfrac{1}{2}[1 - (p_{A,*}^B)^2]e_{A,*}^{1,B} \cdot \frac{p}{p_{A,*}^B} = p_{A,*}^B(r + e_{A,*}^{1,A}) \cdot \frac{p}{p_{A,*}^B} = p(r + e_{A,*}^{1,A}).$$

Then,

$$\begin{aligned}
C &< \phi^{1,B}(p)\left[-\tfrac{1}{2}\left(1-(p_{A,*}^B)^2\right)\left(\frac{p_{A,*}^B - p}{p_{A,*}^B}\phi^{1,B}(0) + \frac{p}{p_{A,*}^B}e_{A,*}^{1,B}\right) + p\left(r + e_{A,*}^{1,A}\right)\right] \\
&= \phi^{1,B}(p)\left[-\tfrac{1}{2}\left(1-(p_{A,*}^B)^2\right)\frac{p_{A,*}^B - p}{p_{A,*}^B}\phi^{1,B}(0) - \tfrac{1}{2}\left(1-(p_{A,*}^B)^2\right)\frac{p}{p_{A,*}^B}e_{A,*}^{1,B} + p\left(r + e_{A,*}^{1,A}\right)\right] \\
&= \phi^{1,B}(p)\left[-\tfrac{1}{2}\left(1-(p_{A,*}^B)^2\right)\frac{p_{A,*}^B - p}{p_{A,*}^B}\phi^{1,B}(0)\right] \\
&\leq -\tfrac{1}{2}[\phi^{1,B}(0)]^2 \cdot \frac{1-(p_{A,*}^B)^2}{p_{A,*}^B} \cdot \delta.
\end{aligned}$$

We conclude that the inequality (22), and thus also inequality (21), can be guaranteed by putting

$$\varepsilon := \frac{\delta}{2(r+2)} \cdot \frac{1-(p_{A,*}^B)^2}{p_{A,*}^B} \cdot [\phi^{1,B}(0)]^2 > 0. \tag{23}$$

Then, $\sigma_A^{0,B}(p) < 0$ whenever $\left|E_A^{0,B}(p) - pE_A^{1,B}(p)\right| < \varepsilon$, as we wanted to prove. □

Proof of Lemma E.7. For $t > 0$, the trajectory satisfies the constraints of Problem 1. We verify that each constraint extends to $t = 0$ by continuity and contradiction arguments.

Condition $\dot{p}^B_{A,0} > 0$: By the ODE (15), $\dot{p}^B_{A,0} = (1 - p^B_{A,0})(e^{0,B}_{A,0} - p^B_{A,0}e^{1,B}_{A,0})$, so it suffices to show that $e^{0,B}_{A,0} - p^B_{A,0}e^{1,B}_{A,0} > 0$. For $t > 0$, the trajectories can be identified with solutions of Problem 1 with initial condition $\hat{p} > p_L$, which satisfy $\dot{p}^B_{A,t} > 0$ and hence $e^{0,B}_{A,t} - p^B_{A,t}e^{1,B}_{A,t} > 0$. By continuity, $e^{0,B}_{A,0} - p^B_{A,0}e^{1,B}_{A,0} \geq 0$. Suppose for contradiction that $e^{0,B}_{A,0} = p^B_{A,0}e^{1,B}_{A,0}$, so that $\dot{p}^B_{A,0} = 0$. By Lemma E.6, $\dot{e}^{0,B}_{A,0} < 0$. Moreover, by Lemma E.5 and $\dot{p}^B_{A,t} > 0$ for $t > 0$, we have $\dot{e}^{1,B}_{A,t} > 0$ for all $t > 0$, so by continuity $\dot{e}^{1,B}_{A,0} \geq 0$. Therefore,

$$\frac{d}{dt}\left(e^{0,B}_{A,t} - p^B_{A,t}e^{1,B}_{A,t}\right)\Big|_{t=0} = \dot{e}^{0,B}_{A,0} - p_L\dot{e}^{1,B}_{A,0} \leq \dot{e}^{0,B}_{A,0} < 0,$$

which contradicts $e^{0,B}_{A,t} - p^B_{A,t}e^{1,B}_{A,t} > 0$ for all $t > 0$. Therefore, $e^{0,B}_{A,0} - p^B_{A,0}e^{1,B}_{A,0} > 0$ and $\dot{p}^B_{A,0} > 0$.

Condition $e^{1,A}_{A,0}, e^{1,B}_{A,0} \in (0, 1)$: By Lemma E.5, $e^{1,A}_{A,t}$ and $e^{1,B}_{A,t}$ are strictly increasing. Therefore, we only need to ensure that $e^{1,A}_{A,0} > 0$ and $e^{1,B}_{A,0} > 0$. Suppose to the contrary that $e^{1,A}_{A,0} = 0$ (the weak version of the inequality $e^{1,A}_{A,0} \geq 0$ follows from the fact that $e^{1,A}_{A,t}$ is part of a solution to Problem 1 with some $\hat{p} \in \mathcal{P}$ for $t > 0$.). Then, by (12),

$$\dot{e}^{1,A}_{A,0} = \tfrac{1}{2}(e^{1,A}_{A,0})^2 - (r + p^B_{A,0}e^{1,B}_{A,0})(1 - e^{1,A}_{A,0}) \leq -r < 0,$$

which is a contradiction with the fact that $\dot{e}^{1,A}_{A,t}$ is continuous and positive for all $t > 0$. Therefore, $e^{1,A}_{A,0} > 0$.

Now suppose to the contrary that $e^{1,B}_{A,0} = 0$. Then, by (13),

$$\dot{e}^{1,B}_{A,0} = \tfrac{1}{2}(e^{1,B}_{A,0})^2 - (r + e^{1,A}_{A,0})(1 - e^{1,B}_{A,0}) \leq -r < 0,$$

which is a contradiction with the fact that $\dot{e}^{1,B}_{A,t}$ is continuous and positive for all $t > 0$. Therefore, $e^{1,B}_{A,0} > 0$.

Condition $e^{0,B}_{A,0} \in (0, 1 - e^{1,B}_{A,0})$: Since $e^{0,B}_{A,t} > 0$ for all $t > 0$, by continuity $e^{0,B}_{A,0} \geq 0$. Suppose for contradiction that $e^{0,B}_{A,0} = 0$. Then, by (14),

$$\dot{e}^{0,B}_{A,0} = -\tfrac{1}{2}(e^{1,B}_{A,0})^2 < 0,$$

which contradicts $e^{0,B}_{A,t} > 0$ for all $t > 0$. Therefore, $e^{0,B}_{A,0} > 0$.

It remains to show that $e^{0,B}_{A,0} < 1 - e^{1,B}_{A,0}$. Since $e^{0,B}_{A,t} + e^{1,B}_{A,t} < 1$ for all $t > 0$, by continuity $e^{0,B}_{A,0} + e^{1,B}_{A,0} \leq 1$. Suppose for contradiction that $e^{0,B}_{A,0} + e^{1,B}_{A,0} = 1$. Then, by (13) and (14),

$$\dot{e}^{0,B}_{A,0} + \dot{e}^{1,B}_{A,0} = \tfrac{1}{2}(e^{0,B}_{A,0})^2 + (r + e^{1,A}_{A,0})[e^{0,B}_{A,0} - (1 - e^{1,B}_{A,0})] = \tfrac{1}{2}(1 - e^{1,B}_{A,0})^2 > 0,$$

which contradicts $e^{0,B}_{A,t} + e^{1,B}_{A,t} < 1$ for all $t > 0$. Therefore, $e^{0,B}_{A,0} < 1 - e^{1,B}_{A,0}$.

Condition $p^B_{A,0} \in (0,1)$: From the initial condition, $p^B_{A,0} = p_L \in (0, p^B_{A,*}) \subset (0,1)$. □

F Supplementary Appendix: Proofs for Appendix D

Proof of Lemma D.4. Consider first the more general, not necessarily symmetric, case. Let $g_M = g^{-j}_{M,t}$. We show that firm j's benefit from postponing revealing by $\Delta t > 0$ is $R^j_t(g_M)\Delta t + o(\Delta t)$, where

$$R^j_t(g_M) = g_M\big[V^{1,B}_A(p^j_{M,t}) - \overline{V}(p^{-j}_t)\big] + \frac{a}{2}\big[E^{1,A}_A(p^{-j}_t)\big]^2 + (e^{0,-j}_{M,t} - p^{-j}_t e^{1,-j}_{M,t})\overline{V}(p^{-j}_t) - (r + e^{0,-j}_{M,t})V^{1,A}_A(p^{-j}_t), \tag{24}$$

where $\overline{V}(p^{-j}_t) = V^{1,A}_A(p^{-j}_t) + (1 - p^{-j}_t)(V^{1,A}_A)'(p^{-j}_t)$.[6] To see this, consider firm j that is successful at time t. Revealing at time t gives firm j payoff $V^{1,A}_A(p^{-j}_t)$. Revealing at time $t+\Delta t$, assuming that the rival $-j$ reveals a breakthrough with the hazard rate g_M, gives firm j payoff

$$\frac{a}{2}\big[E^{1,A}_A(p^{-j}_t)\big]^2\Delta t + g_M\Delta t \cdot V^{1,B}_A(p^j_{M,t}) + \big[1 - (r + g_M + p^{-j}_t e^{1,-j}_{M,t})\Delta t\big]V^{1,A}_A(p^{-j}_{t+\Delta t}).$$

Then the gain from postponing revealing by Δt divided by Δt as $\Delta t \searrow 0$ is

$$R^j_t(g_M) = \frac{a}{2}\big[E^{1,A}_A(p^{-j}_t)\big]^2 + g_M[V^{1,B}_A(p^j_{M,t}) - V^{1,A}_A(p^{-j}_t)] - (r + p^{-j}_t e^{1,-j}_{M,t})V^{1,A}_A(p^{-j}_t) + (V^{1,A}_A)'(p^{-j}_t)\dot{p}^{-j}_{M,t}.$$

Substituting in for $\dot{p}^{-j}_{M,t}$ using the law of motion from Lemma 2, we obtain (24).

Note that when the coefficient at g_M is positive, then the hazard rates of revealing of the two firms are strategic substitutes: Firm j prefers to postpone revealing whenever $g^{-j}_{M,t}$ is above the threshold $\bar{g}^{-j}_{M,t} = -R^j_t(0)/[V^{1,B}_A(p^j_{M,t}) - \overline{V}(p^{-j}_t)]$. As a result, $g^j_{M,t} = 0$ if $g^{-j}_{M,t}$ is above the threshold (i.e., $g^{-j}_{M,t} > \bar{g}^{-j}_{M,t}$); $g^j_{M,t} = e^{0,j}_{M,t}$ (or $+\infty$ when $p^j_{M,t} > 0$) if $g^{-j}_{M,t}$ is below the threshold (i.e., $g^{-j}_{M,t} < \bar{g}^{-j}_{M,t}$); and $g^j_{M,t}$ is arbitrary when $g^{-j}_{M,t}$ equals the threshold (i.e., $g^{-j}_{M,t} = \bar{g}^{-j}_{M,t}$).

In the remainder of the proof consider the symmetric case (dropping the superscript j). In such a case it can be shown numerically that $V^{1,B}_A(p) - \overline{V}(p)$ is indeed positive for all $p \in [0,1)$. Thus, if the firms have the same posterior beliefs about each other's success, then the coefficient of g_M in $R_t(g_M)$ is positive for any $t \geq 0$. Intuitively, it is a

[6]We denote $(V^{1,A}_A)'$ the derivative of the function $v^{1,A}$ with respect to the second argument.

consequence of informational advantage a firm gets from knowing rival's state as stated in Proposition 6.

We now proceed with the proofs of properties (i)–(iv).

(i) Suppose to the contrary that $\lim_{s \nearrow t} g_{M,s} = 0$ and $g_{M,t} > 0$. Then firm j's incentive to postpone revealing at time t is $R_t(g_{M,t}) = 0$. Since the coefficient of g in $R_t(g)$ is positive, $R_t(0) < 0$. By continuity of $R_s(0)$ in s, there exists $\Delta t > 0$ such that $R_s(0) < 0$ for all $s \in [t - \Delta t, t]$. Pick any such s at which $g_{M,s} = 0$. Then $R_s(g_{M,s}) = R_s(0) < 0$, so firms have a strict incentive to reveal at time s, contradicting $g_{M,s} = 0$.

(ii) Suppose to the contrary that $g_{M,t} > 0$ for an infinite increasing time sequence $t = t_1, t_2, t_3, \ldots$ such that $t_n \to +\infty$ as $n \to +\infty$. Since $p_{M,t}$ is non-decreasing, $p_{M,t}$ converges to some value $p_{M,\infty}$ as $t \to +\infty$. By assumption, $G_M(p)$ has at most a finite number of points of discontinuities, and so $G_M(p_{M,t})$ converges to some $g_\infty \geq 0$. Then $e^1_{M,t}$ has to converge to the solution of the steady-state version of the equation (14), $e^0_{M,t}$ to a steady-state version of the equation (15), and these limits have to satisfy the steady-state version of equation (16). Moreover, $e^1_{M,t_n} = E^{1,A}_A(p_{t_n})$ converges to $E^{1,A}_A(p_\infty)$. However, it can be shown numerically that the critical point of the system of ODEs (14)–(16) with $E^1_M(p_{M,t}) = E^{1,A}_A(p_{M,t})$ has at most one critical point, and this critical point is of a source type (all the eigenvalues of the Jacobian have positive real parts). This is a contradiction, as the system of ODEs cannot converge to a critical point of a source type. The situation is illustrated in Figure 4.

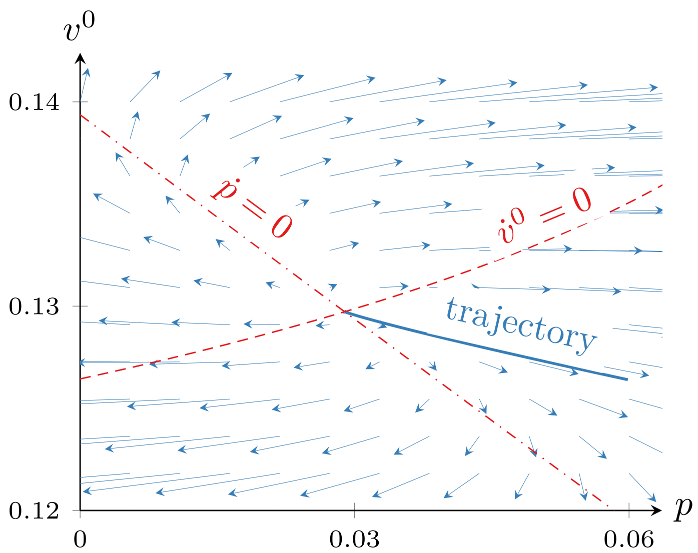


Figure 4: Phase diagram for the system of ODEs (15)–(16) for $\rho = 0.2$. It demonstrates that the critical point is of a source type.

(iii) Firm j's continuation value at time T is $v^1_{M,T} = V^1_S(p_{M,T})$. Since firm j is revealing prior to time T, we have $\lim_{t \nearrow T} v^1_{M,t} = V^{1,A}_A(p_{M,T})$. Firm's continuation value $v^1_{M,t}$ has to be continuous in t, as it is associated to a fixed state, so

$$V^1_S(p_{M,T}) = v^1_{M,T} = \lim_{t \nearrow T} v^1_{M,t} = V^{1,A}_A(p_{M,T}).$$

(iv) Recall that we denote $\overline{p}_M$ the unique value of belief such that $V^1_S(\overline{p}_M) = V^{1,A}_A(\overline{p}_M)$; see Lemma D.3. The system of ODEs (14)–(16) with $E^1_M(p_{M,t}) = E^{1,A}_A(p_{M,t})$ can be solved numerically going from the posterior $p = \overline{p}_M$ back to $p = 0$. The numerical solution demonstrates that $G_M(p)$ is strictly decreasing in p for $p \in [0, \overline{p}_M]$, whenever the solution exists. Claim (i) guarantees us then that going back in time firms cannot stop revealing spontaneously (whilst $g_{M,t}$ can discretely drop to 0 as t increases, it cannot happen as t decreases). □